Lecture 6
Solution Of Bi-quadratic Equations By Using Descartes Method

Let the given Bi-quadratic equation be

$$x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

Now to remove the second term, we have to diminish the roots by

$$h = -\frac{a_1}{a_0}$$

$\therefore$ $y = x - h = x + \dfrac{a_1}{a_0}$.

or $x = y - \dfrac{a_1}{a_0}$. Substituting the value of x in the given equation, the transformed equation becomes

$$a_0^4y^4 + 6a_0^2y^2(a_0a_1 - a_1^2) + 4a_0y(2a_1^2 - 3a_0a_1a_2 + a_0^2a_3) + (6a_0a_2a_1^2 - 3a_1^4 - 4a_0^2a_1a_3 + a_0^3a_4) = 0$$

$$a_0^4y^4 + 6a_0y^2(a_0a_1 - a_1^2) + 4a_0y(2a_1^2 - 3a_0a_1a_2 + a_0^2a_3) + [a_0^2(3a_2^2 - 4a_1a_3 + a_0a_4) - 3(a_0a_2 - a_1^2)^2] = 0$$

Putting $y = \dfrac{z}{a_0}, H = a_0a_2 - a_1^2, G = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_2$ and $3a_2^2 - 4a_1a_3 + a_0a_4$, we get the transformed equation as

$$z^4 + 6Hz^2 + 4Gz + (a_0^2I - 3H^2) = 0, \tag{0.1}$$

where $a_0x + a_1 = z$.

Let the quadratic factor of (0.1) be

$$z^2 + pz + q \qquad \text{and} \qquad z^2 - pz + q',$$

so that

$$(z^2 + pz + q)(z^2 - pz + q') = 0 \tag{0.2}$$

is the same as (0.1).

Comparing the coefficients of the left hand side of (0.1) and (0.2), we get

$$q' + q - p^2 = 6H$$

$$\text{or} \qquad q' + q = 6H + p^2 \tag{0.3}$$

$$p(q' - q) = 4G$$

$$\text{or} \qquad q' - q = \frac{4G}{p} \tag{0.4}$$

Also, $qq' = a_0^2I - 3H^2$ or $4qq' = 4a_0^2I - 12H^2$.

Since, we have

$$4qq' = (q' + q)^2 - (q' - q)^2 \tag{0.5}$$

$$\therefore \qquad 4a_0^2I - 12H^2 = p^4 + 12p^2H + 36H^2 - \frac{16G^2}{p^2}$$

$$\text{or} \qquad p^6 + 12p^4H + (48H^2 - 4a_0^2I)p^2 - 16G^2 = 0$$

Putting $p^2 = t$, we get

$$t^3 + 12Ht^2 + 4(12H^2 - a_0^2I)t - 16G^2 = 0 \tag{0.6}$$

Equation (0.6) is the called Descartes Resolvent. Now, we can find t from (0.6) and so also p.
From (0.3) and (0.4), we can find q and q'. Thus, the two quadratic equations $z^2 + pz + q = 0$ and $z^2 - pz + q' = 0$ can be solved.
Let z_1, z_2, z_3, z_4 be the roots of these equations, then corresponding to these we can find the four values of x from the relation

$$a_0 x + a_1 = z.$$

Example 1. *Solve*

$$x^4 - 6x^2 + 8x - 3 = 0. \tag{0.7}$$

Sol. Let the quadratic factors be

$$(x^2 + px + q)(x^2 - px + q') = 0 \tag{0.8}$$

$$\implies \quad x^4 - px^3 + q'x^2 + p^2x^2 + pq'x + qx^2 - pqx + qq' = 0$$

$$\implies \quad x^4 + (p + q' - p^2)x^2 + p(q' - q)x + qq' = 0 \tag{0.9}$$

Comparing the coefficients of the left hand side of (0.7) and (0.9), we get

$$q' + q - p^2 = -6$$

$$\text{or} \qquad q' + q = p^2 - 6 \tag{0.10}$$

$$p(q' - q) = 8$$

$$\text{or} \qquad q' - q = \frac{8}{p} \tag{0.11}$$

Also, $qq' = -3$
Since, we have

$$(q' + q)^2 - (q' - q)^2 = 4qq'$$

$$(p^2 - 6)^2 - \left(\frac{8}{p}\right)^2 = -12$$

$$\implies \quad p^6 + 36p^2 - 12p^4 - 64 + 12p^2 = 0$$

$$\implies \quad p^6 - 12p^4 + +48p^2 - 64 = 0$$

$$\implies \quad (p^2)^3 - 12(p^2)^2 + 48(p^2) - 64 = 0$$

Put $p^2 = t$, we get

$$t^3 - 12t^2 + 48t - 64 = 0 \tag{0.12}$$

test the cubic for numbers which are perfect squares i,e. $t = 1, 4, 9, 16, 25, \cdots$
$\therefore \quad t = 4$, satisfies equation (0.12).
So that $p^2 = t = 4 \implies \quad p^2 = 4 \quad \implies \quad p = 2.$
Also, from (0.11), we have

$$q' - q = \frac{8}{2}$$

$$q' - q = 4 \tag{0.13}$$

From (0.10), we get

$$\begin{aligned} q' + q &= 4 - 6 \\ \implies \quad q' + q &= -2 \end{aligned} \tag{0.14}$$

Adding (0.13) and (0.14), we get

$$\begin{aligned} 2q' &= 2 \\ \implies \quad q' &= 1 \end{aligned}$$

Put $q' = 1$ in (0.14), we have

$$\begin{aligned} q &= -2 - 1 \\ \implies \quad q &= -3. \end{aligned}$$

Thus substitute p, q, q' in (0.8), we get

$$\begin{aligned} (x^2 + 2x - 3)(x^2 - 2x + 1) &= 0 \\ \implies \quad (x^2 + 2x - 3) = 0 \quad \text{or} \quad (x^2 - 2x + 1) &= 0 \\ \implies \quad (x + 3)(x - 1) = 0 \quad \text{or} \quad (x - 1)(x - 1) &= 0 \\ \implies \quad x = 1, -3 \quad \text{or} \quad x &= 1, 1 \end{aligned}$$

Hence the four roots are $x = 1, 1, 1, -3$.

Example 2. *Solve* $x^4 - 4x^3 + 6x^2 + x - 10 = 0$.

Sol. we first remove the second term, so we diminish roots of the given equation by h, where

$$h = \frac{\text{Sum of roots}}{\text{Degree of equation}} = \frac{-\frac{-4}{1}}{4} = 1$$

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 6 & 1 & -10 \\ \hline & & 1 & -3 & 3 & 4 \\ & 1 & -3 & 3 & 4 & |\underline{-6} \\ \hline & & 1 & -2 & 1 & \\ & 1 & -2 & 1 & |\underline{5} & \\ \hline & & 1 & -1 & & \\ & 1 & -1 & |\underline{0} & & \\ \hline & & 1 & & & \\ & 1 & |\underline{0} & & & \\ & 1 & & & & \end{array}$$

$\therefore$ the transformed equation is

$$y^4 + 5y - 6 = 0 \tag{0.15}$$

Let the quadratic factors of the above equation be

$$(y^2 + py + q)(y^2 - py + q') = 0 \tag{0.16}$$

Comparing the coefficients of the left hand side of (0.15) and (0.16), we get

$$q' + q - p^2 = 0$$

$$\text{or} \qquad q' + q = p^2 \tag{0.17}$$

$$p(q' - q) = 5$$

$$\text{or} \qquad q' - q = \frac{5}{p} \tag{0.18}$$

Also, $qq' = -6$
Since, we have

$$(q' + q)^2 - (q' - q)^2 = 4qq'$$

$$(p^2)^2 - \left(\frac{5}{p}\right)^2 = -24$$

$$\implies \quad p^6 + 24p^2 - 25 = 0$$

$$\implies \quad (p^2)^3 + 24(p^2) - 25 = 0$$

Put $p^2 = t$, we get

$$t^3 + 24t^2 - 25 = 0 \tag{0.19}$$

test the cubic for numbers which are perfect squares i,e. $t = 1, 4, 9, 16, 25, \cdots$
$\therefore \quad t = 1$, satisfies equation (0.19).
So that $p^2 = t = 1 \implies \quad p^2 = 1 \quad \implies \quad p = 1$.
Also, from (0.17), we have

$$q' + q = 1 \tag{0.20}$$

$$q' - q = \frac{5}{1}$$

$$q' - q = 5 \tag{0.21}$$

Adding (0.20) and (0.21), we get

$$q' = 3$$

Put $q' = 3$ in (0.21), we have

$$q = -2$$

Thus substitute p, q, q' in (0.16), we get

$$(y^2 + y - 2)(y^2 - y + 3) = 0$$

Which gives $y = 1, 2, \dfrac{1 \pm i\sqrt{11}}{2}$.
Since $x = y + 1$, gives $x = 2, -1, 1, \dfrac{3 \pm i\sqrt{11}}{2}$ are the required roots.

Lecture 5
SOLUTION OF CUBIC EQUATIONS BY USING CARDEN'S METHOD

Let the cubic equation be

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{0.1}$$

We first remove the second term of equation (0.1), so setting $h = -\dfrac{a_1}{a_0}$.

Put $y = x - h$, gives $y = x + \dfrac{a_1}{a_0}$, or $x = y - \dfrac{a_1}{a_0}$.

Substituting this value of x in (0.1), we get

$$a_0\left(y - \frac{a_1}{a_0}\right)^3 + 3a_1\left(y - \frac{a_1}{a_0}\right)^2 + 3a_2\left(y - \frac{a_1}{a_0}\right) + a_3 = 0$$

$$\text{or} \qquad a_0^3y^3 + 3a_0y(a_0a_2 - a_1^2) + (2a_1^3 - 3a_0a_1a_2 + a_0^2a_3) = 0$$

Putting $y = \dfrac{z}{a_0}$ in this equation, we get

$$z^3 + 3z(a_0a_2 - a_1^2) + (2a_1^3 - 3a_0a_1a_2 + a_0^2a_3) = 0$$

$$\text{or} \qquad z^3 + 3Hz + G = 0 \tag{0.2}$$

where $H = a_0a_2 - a_1^2$ and $G = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_3$. Now to solve (0.2), we put

$$z = u + v.$$

Cubing it, we get

$$z^3 = (u + v)^3$$

$$\text{or} \qquad z^3 - 3uvz - (u^3 + v^3) = 0. \tag{0.3}$$

On equating coefficients of z and constant terms of (0.2) and (0.3), we have

$$u^3 + v^3 = -G$$

$$\text{and} \qquad uv = -H$$

$$\therefore \qquad u^3v^3 = -H^3.$$

Now a quadratic equation in t whose roots are u^3 and v^3 is

$$t^2 - (u^3 - v^3)t + u^3v^3 = 0$$

$$\text{or} \qquad t^2 - Gt + -H^3 = 0.$$

This gives

$$t = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2},$$

which means that

$$u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2}$$

and

$$v^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2}.$$

$$\therefore \qquad u = \left[\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}, \left[\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega, \left[\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega^2$$

$$\text{and} \qquad v = \left[\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}, \left[\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega, \left[\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega^2,$$

where ω, ω^2 the complex roots of unity.

Putting $\left[\dfrac{-G+\sqrt{G^2+4H^3}}{2}\right]^{\frac{1}{3}} = \alpha$ and $\left[\dfrac{-G-\sqrt{G^2+4H^3}}{2}\right]^{\frac{1}{3}} = \beta$, we obtain three values of z as

$$\alpha+\beta, \omega\alpha+\omega^2\beta, \omega^2\alpha+\omega\beta$$

and then the values of x can be obtained from

$$z = a_0x + a_1 \qquad \text{or} \qquad x = \frac{z-a_1}{a_0}.$$

Example 1. *Solve the cubic equation $x^3-6x-9=0$, by Carden's method.*

Sol. The given cubic equation is

$$x^3-6x-9=0. \tag{0.4}$$

Here 2^{nd} term is missing. Put $x = u+v \implies x^3 = (u+v)^3 = u^3+v^3+3uv(u+v)$

$$\implies x^3-3uvx-(u^3+v^3)=0 \tag{0.5}$$

Equating coefficients of (0.4) and (0.5), we get
$u^3+v^3=9$ and $3uv=6 \implies uv=2 \implies u^3v^3=8$.
Now take u^3 and v^3 as roots of quadratic in t, we have

$$t^2-(u^3+v^3)t+u^3v^3=0$$
$$\implies t^2-9t+8=0$$
$$\implies (t-1)(t-8)=0$$
$$\implies t=1,8$$
$$\therefore \quad u^3=1 \quad \text{and} \quad v^3=8$$
$$\implies u=1 \quad \text{and} \quad v=2$$
$$\therefore \quad x=1+2=3,$$

is the one root. Now using synthetic division we have

3 |	1	0	−6	−9
		3	9	9
	1	3	3	|0

$\therefore \quad x^2+3x+3=0$

$\implies x = \dfrac{-3\pm\sqrt{9-12}}{2}$

$\implies x = \dfrac{-3\pm\sqrt{-3}}{2}$

$\implies x = \dfrac{-3\pm i\sqrt{3}}{2}$

Hence the roots are $x = 3, \dfrac{-3\pm i\sqrt{3}}{2}$

Example 2. *Solve the cubic equation $x^3+3x^2+12x-16=0$.*

Sol. Here second term is present, so we first remove this second term and hence diminish it by h where

$$h = \frac{\text{Sum of roots}}{\text{Degree of equation}} = \frac{-\frac{3}{1}}{3} = -1$$

$$\begin{array}{r|rrrr}
-1 & 1 & 3 & 12 & -16 \\
 & & -1 & -2 & -10 \\
\hline
 & 1 & 2 & 10 & \underline{|-26} \\
 & & -1 & -1 & \\
\hline
 & 1 & 1 & \underline{|9} & \\
 & & -1 & & \\
\hline
 & 1 & \underline{|0} & &
\end{array}$$

$\therefore$ the transformed equation is

$$y^3 + 9y - 26 = 0 \tag{0.6}$$

Put $y = u + v$

$$\implies \quad y^3 - 3uvy - (u^3 + v^3) = 0 \tag{0.7}$$

Equating coefficients of (0.6) and (0.7), we get $u^3 + v^3 = 26$ and $3uv = -9 \implies uv = -3 \implies u^3v^3 = -27$.

Now take u^3 and v^3 as roots of quadratic in t, we have

$$t^2 - (u^3 + v^3)t + u^3v^3 = 0$$

$$\implies \quad t^2 - 26t - 27 = 0$$

$$\implies \quad (t+1)(t-27) = 0$$

$$\implies t = -1, 27$$

$$\therefore \quad u^3 = -1 \quad \text{and} \quad v^3 = 27$$

$$\implies \quad u = -1 \quad \text{and} \quad v = 3$$

$$\therefore \quad y = -1 + 3 = 2,$$

is the one root. Now using synthetic division we have

$$\begin{array}{r|rrrr}
2 & 1 & 0 & 9 & -26 \\
 & & 2 & 4 & 26 \\
\hline
 & 1 & 2 & 13 & \underline{|0}
\end{array}$$

$\therefore \quad y^2 + 2y + 13 = 0$

$\implies \quad y = \dfrac{-2 \pm \sqrt{4 - 52}}{2}$

$\implies y = \dfrac{-2 \pm \sqrt{-48}}{2}$

$\implies y = -1 \pm 2i\sqrt{3}$

Since $\quad y = x - h \quad \implies \quad x = y + h = y - 1$

$\therefore \quad x = (2-1), -1 \pm 2i\sqrt{3} - 1$

$\implies \quad x = 1, -2 \pm 2i\sqrt{3}.$

Hence the roots are $x = 1, -2 \pm 2i\sqrt{3}$.

Nature Of The Roots Of The Cubic Equation

$$z^3 + 3Hz + G = 0$$

We know that the roots of the above cubic equation are given by $z = x + y$, where u^3 and v^3 are the roots of the quadratic equation $t^2 + Gt + H = 0$. The discriminant of this quadratic equation is $G^2 + 4H^3$. Therefore, we have the following three cases.

(i). $G^2 + 4H^3 > 0$.

In this case both u^3 and v^3 are real, so u and v are also real. Therefore, the three values of z are $u + v, u\omega + v\omega^2, u\omega^2 + v\omega$. The first root is real and the other two roots are complex.

Hence in this case one root is real and other two are complex.

(ii). $G^2 + 4H^3 = 0$.

In this case both u^3 and v^3 are real and equal. Hence, the three roots of the cubic equation are

$$\begin{aligned} u + v &= u + u = 2u \\ u\omega + v\omega^2 &= u(\omega + \omega^2) = -u \\ u\omega^2 + v\omega^2 &= u(\omega^2 + \omega) = -u. \end{aligned}$$

Thus in this case, all the roots are real and rational but two of them are equal.

(iii). $G^2 + 4H^3 < 0$.

In this case u^3 and v^3 are conjugate quantities and as such u and v are also conjugate complex quantities of the $a \pm ib$, where a and b are real.

Hence, the roots of the cubic equation are

$$\begin{aligned} u + v &= (a + ib) + (a - ib) = 2a \\ u\omega + v\omega^2 &= (a + ib)\omega + (a - ib)\omega^2 \\ &= a(\omega + \omega^2) + ib(\omega - \omega^2) \\ &= -a - b\sqrt{3} \qquad \text{using } 1 + \omega + \omega^2 = 0 \\ \text{and} \qquad u\omega^2 + v\omega^2 &= (a + ib)\omega^2 + (a - ib)\omega \\ &= a(\omega^2 + \omega) + ib(\omega^2 - \omega) \\ &= -a + b\sqrt{3} \qquad \text{using } 1 + \omega + \omega^2 = 0. \end{aligned}$$

Thus, in this case all the three roots are real and distinct but unequal.

Example 3. *Discuss nature of roots of the equation* $x^3 + 15x - 124 = 0$.

Sol. Comparing it with

$$z^3 + 3Hz + G = 0,$$

we get

$$H = 5, \quad G = -124 \qquad \therefore \quad G^2 + 4H^3 = (-124)^2 + 4(125) > 0.$$

By case (i), the given equation has one real and two complex roots.

Example 4. *Discuss nature of roots of the equation* $x^3 + 3x - 14 = 0$.

Sol. Comparing it with

$$z^3 + 3Hz + G = 0,$$

we get

$$H = 1, \quad G = -14 \qquad \therefore \quad G^2 + 4H^3 = (-14)^2 + 4(1) > 0.$$

Again by case (i), the given equation has one real and two complex roots.

Unit-I: Differential Equations

By Aijaz A. Najar

LECTURE-1

1 Differential Equations:

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.
Example 1.1: The equations

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = sint \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} = v \tag{3}$$

$$\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} + \frac{\partial^2 v}{\partial^2 z} = 0 \tag{4}$$

are all examples of differential equations.
From the brief list of differential equations in Example 1.1, it is clear that the various variables and derivatives involved in a differential equation can occur in a variety of ways. Clearly some kind of classification must be made.

To begin with, we classify differential equations according to whether there is one or more than one independent variable involved.

Definition 1: A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.
In Example 1.1, Equations (1) and (2) are ordinary differential equations. In Equation (1) the variable x is the single independent variable, and is a dependent variable. In Equation (2) the independent variable is t, whereas x is dependent.

Definition 2: A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.
For Example, Equations (3) and (4) in the Example 1.1 are partial differential equations. In Equation (3) the variables s and t are independent variables and v is a dependent variable. In Equation (4) there are three independent variables: x, y, and z in this equation v is dependent.
We further classify differential equations, both ordinary and partial, according to the order of the highest derivative appearing in the equation. For this purpose we give the following definition.

Definition 3: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.
The ordinary differential equation (1) is of the second order, since the highest derivative involved is a second derivative. Equation (2) is an ordinary differential equation of the fourth order. The partial differential equations (3) and (4) are of the first and second orders respectively.

Definition 4: A linear ordinary differential equation of order n in the dependent variable and the independent variable x is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x) \tag{5}$$

where a_0 is not identically zero.
From the above definition, it is clear

(a) that the dependent variable and its various derivatives occur to the first degree only
(b) that no products of and/or any of its derivatives are present and
(c) that no transcendental functions of and/or its derivatives occur.
Example 1.2: The following ordinary differential equations are both linear. In each case y is the dependent variable. Observe that and its various derivatives occur to the first degree only and that no products of and/or any of its derivatives are present.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) + 6y = 0 \tag{6}$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x \tag{7}$$

Definition 5: A non linear ordinary differential equation is an ordinary differential equation that is not linear.
Example 1.3: The following ordinary differential equations are all nonlinear:

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) + 6y^2 = 0 \tag{8}$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 6y = 0 \tag{9}$$

$$\frac{d^2y}{dx^2} + 5y\left(\frac{dy}{dx}\right) + 6y = 0 \tag{10}$$

1.1 Origin and Application of Differential Equations:

Having classified differential equations in various ways, let us now consider briefly where, and how, such equations actually originate. In this way we shall obtain some indication of the great variety of subjects to which the theory and methods of differential equations may be applied. Differential equations occur in connection with numerous problems that are encountered in the various branches of science and engineering. We indicate a few such problems in the following list, which could easily be extended to fill many pages.
1. The problem of determining the motion of a projectile, rocket, satellite, or planet.

2. The problem of determining the charge or current in an electric circuit.
3. The problem of the conduction of heat in a rod or in a slab.
4. The problem of determining the vibrations of a wire or a membrane.
5. The study of the rate of decomposition of a radioactive substance or the rate of growth of a population.
6. The study of the reactions of chemicals.
7. The problem of the determination of curves that have certain geometrical properties.

The mathematical formulation of such problems give rise to differential equations. But just how does this occur? In the situations under consideration in each of the above problems the objects involved obey certain scientific laws. These laws involve various rates of change of one or more quantities with respect to other quantities. Let us re- call that such rates of change are expressed mathematically by derivatives. In the mathematical formulation of each of the above situations, the various rates of change are thus expressed by various derivatives and the scientific laws themselves become mathematical equations involving derivatives, that is, differential equations.

LECTURE-2

2 Exact Differential Equations and integrating factors:

The first-order differential equations to be studied in this chapter may be expressed in either the derivative form

$$\frac{dy}{dx} = f(x, y) \tag{11}$$

or the differential form

$$M(x, y)dx + N(x, y)dy = 0 \tag{12}$$

An equation in one of these forms may readily be written in the other form.
Definition 6: Let F be a function of two real variables such that F has

continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy \quad forall \quad (x,y)\epsilon D \tag{13}$$

Definition 7: The expression

$$M(x,y)dx + N(x,y)dy \tag{14}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all $(x,)\epsilon D$. That is, expression (14) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x}dx = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y}dy = N(x,y) \tag{15}$$

If $M(x,y)dx+N(x,y)dy$ is an exact differential, then the differential equation $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation.

Rule for solving $Mdx + Ndx = 0$ when it is exact:

1. First integrate w.r.t. y the terms in Mdx but treating y as a constant.
2. Then integrate w.r.t. y only those terms of Ndy which do not contain x.
3. Equate to some constant the sum of the results of (1) and (2).

That is, $\int Mdx + \int Ndy = c$ 4. If N has no term which is free from x, then $\int Mdx = c$, y constant.

Example 2.1: Solve the differential equation

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$$

Solution: Here

$$\frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$$

Therefore, the equation is exact. Therefore, the solution is

$$\begin{aligned} \int\limits_{y\ constant} (x^3 + 3xy^2)dx + \int y^3 dy &= c \\ x^4 + y^4 + 6x^2y^2 &= k \end{aligned}$$

LECTURE-3

Theorem 1 (Necessary and Sufficient Condition for Exactness): Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{16}$$

where M and N have continuous first partial derivatives at all points (x, y) in a rectangular domain D. If the differential equation (16) is exact in D, then

$$\frac{\partial M(x,y)}{\partial x} = \frac{\partial N(x,y)}{\partial y} \quad for\ all\ \ (x,y)\ \epsilon\ D \tag{17}$$

Conversely, if

$$\frac{\partial M(x,y)}{\partial x} = \frac{\partial N(x,y)}{\partial y} \quad for\ all\ \ (x,y)\ \epsilon\ D$$

then the differential equation (16) is exact in D.

Proof: Necessary Part: If the differential equation (16) is exact in D, then$Mdx + Ndy$ is an exact differential in D. By definition of an exact differential, there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x}dx = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y}dy = N(x,y)$$

for all $(x, y)\epsilon D$. Then

$$\frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial M(x,y)}{\partial y} \quad \text{and} \quad \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$$

Now, using the continuity of the partial derivatives, we have

$$\frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial^2 F(x,y)}{\partial y \partial x}$$

and therefore

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Sufficient Part: we start with the hypothesis that

$$\frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial^2 F(x,y)}{\partial y \partial x}$$

for all $(x, y) \epsilon D$, and show that $dx + Ndy = 0$ is exact in D. This means that we must prove that there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \tag{18}$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) \tag{19}$$

for all $(x, y) \epsilon D$. We can certainly find some $F(x, y)$ satisfying either (18) or (19), but what about both? Let us assume that F satisfies (18) and proceed. Then

$$F(x,y) = \int M(x,y)\partial x + \phi(y) \tag{20}$$

where $\int M(x, y)\partial x$ indicates a partial integration with respect to x, holding y constant and ϕ is an arbitrary function of y only. This $\phi(y)$ is needed in (20) so that $F(x, y)$ given by (20) will represent all solutions of (18). It corresponds to a constant of integration in the "one-variable" case. Differentiating (20) partially with respect to y, we obtain

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\partial x + \frac{d\phi(y)}{dy}$$

Now if (19) is to be satisfied, we must have

$$N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\partial x + \frac{d\phi(y)}{dy} \tag{21}$$

and hence

$$\frac{d\phi(y)}{dy} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x \tag{22}$$

Since ϕ is a function of y only, the derivative $\frac{d\phi}{dy}$ must also be independent of x. That is, in order for (21) to hold,

$$N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x \tag{23}$$

must be independent of x. We show

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\right] = 0$$

we have

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\right] = \frac{\partial N(x,y)}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int M(x,y)\partial x$$

and hence

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\right] = \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}$$

But by hypothesis,

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial M(x,y)}{\partial y}$$

Thus,

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\right] = 0$$

Hence, we can write

$$\phi(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\right]\partial y$$

Substituting this into Equation (20), we have

$$F(x,y) = \int M(x,y)\partial x + \int \left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\right]\partial y \tag{24}$$

This $F(x,y)$ thus satisfies both (18) and (19) for all $(x,y)\epsilon D$, and so $Mdx + Ndy = 0$ is exact in D.

Example 2.2: consider the equation $y^2dx+2xydy = 0$, we have $M(x, y) = y^2$ and $N(x, y) = 2xy$. Also

$$\frac{\partial N(x,y)}{\partial x} = 2y = \frac{\partial M(x,y)}{\partial y}$$

Thus the given differential equation is exact in any rectangular region D.
Now, Consider $ydx+2xdy = 0$, we have $M(x, y) = y$ and $N(x, y) = 2x$. Also

$$\frac{\partial N(x,y)}{\partial x} = 2 \neq 1 = \frac{\partial M(x,y)}{\partial y}$$

Therefore, this second differential equation is not exact in any rectangular region D.

LECTURE-4

3 Integrating factor:

If the differential equation $M(x, y)dx+N(x, y)dy = 0$ is not exact in a domain D but the differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact, then $\mu(x, y)$ is called an integrating factor of the differential equation $M(x, y)dx + N(x, y)dy = 0$.
In Example 2.1, the second differential equation was not exact, but by multiplying it with y becomes an exact equation in any rectangular domain. Thus y is the integrating factor of the differential equation $ydx + 2xdy = 0$

3.1 Some rules for finding integrating factors of the equation $Mdx + Ndy = 0$ to make it exact

Rule-1: If $Mdx + Ndy \neq 0$ and the equation is homogeneous, then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$
Rule-2: If $Mx + Ny \neq 0$ is not exact, but is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor of $Mdx + Ndy = 0$

provided $Mx - Ny \neq 0$.
Example 3.1:

$$(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$$

The equation is not exact. but the given equation is homogeneous and by Rule-1, $\frac{1}{Mx+Ny}$ is the integrating factor.
Now

$$Mx = x^3y - 2x^2y^2, Ny = -x^3y + 3x^2y^2.$$

Therefore, the integrating factor is

$$I.F = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

Multiplying the given equation by the I.F., we have

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx + \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 0$$

This equation is now exact because,

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}$$

Example 3.2:

$$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$$

The equation is of the form

$$yf_1(xy)ydx + xf_2(xy)xdy = 0$$

Now

$$Mx = xy(xy + 2x^2y^2), Ny = yx(xy - x^2y^2).$$

Therefore, the integrating factor is

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying the given equation by the I.F., we have

$$\frac{y(xy+2x^2y^2)}{3x^3y^3}dx + \frac{x(xy-x^2y^2)}{3x^3y^3}dy = 0$$

This equation is now exact because,

$$\frac{\partial M}{\partial y} = -\frac{-1}{3x^2y^2} = \frac{\partial N}{\partial x}$$

Rule-3: When $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x alone say $f(x)$, then $I.F = e^{\int f(x)dx}$
Rule-4: When $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y alone say $f(y)$, then $I.F = e^{\int f(y)dy}$
Example 3.3: Consider the differential equation $(x^2 + y^2 + 2x)dx + 2ydy$.
Here

$$\frac{\partial M}{\partial y} = 2y, \text{ and } \frac{\partial N}{\partial x} = 0$$

But

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-0}{2y} = 1$$

which may be regarded as a function of x.
Therefore, $I.F. = e^{\int 1dx} = e^x$.
Exercises for the reader:
1. $(x^3 + xy^4)dx + 2y^3dy = 0$
2. $(x^2 + y^2)dx - 2xydy = 0$
3. $(xy^3 + y)dx + 2(xy^2 + x + y^4)dy = 0$

LECTURE- 5

4 Linear and Bernoulli Equation:

Definition 8:A first-order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + p(x)y = Q(x) \tag{25}$$

For example, the equation $\frac{dy}{dx}+(1+\frac{1}{x})y = x^2$ is a first-order linear differential equation.
The integrating factor for the linear differential equation

$$\frac{dy}{dx} + p(x)y = Q(x)$$

is given by

$$\mu(x,y) = e^{\int p(x)dx} \tag{26}$$

Example 4.1: Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Then, the integrating factor is $\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{logx} = x$

4.1 Bernoulli equation:

Definition 9: An equation of the form

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \tag{27}$$

is called a Bernoulli differential equation.

We observe that if $n = 0$ or 1, then the Bernoulli equation (27) is actually a linear equation and is therefore readily solvable as such. However, in the general case in which $n = 0$ or 1, this simple situation does not hold and we must proceed in a different manner. We now state and prove Theorem 2, which gives a method of solution in the general case.
Theorem 2: Suppose $n \neq 0$ or 1. Then the transformation $v = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + p(x)y = Q(x)y^n$$

to a linear equation in v.
proof: We first multiply Equation (27) by y^{-n}, thereby expressing it in the equivalent form

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = Q(x) \tag{28}$$

If we let, $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

and equation (28) reduces to

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)v = Q(x)$$

Or

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)Q(x)$$

Let, $(1-n)p(x) = p_1(x)$ and $(1-n)Q(x) = Q_1(x)$. Then, we have

$$\frac{dv}{dx} + p_1(x)v = Q_1(x)$$

which is linear in v.
Example 4.2: Consider the Bernoulli differential equation

$$\frac{dy}{dx} + y = xy^3$$

Here $n = 3$, We first multiply the equation throughout by y^{-3}, thereby expressing it in the equivalent form

$$y^{-3}\frac{dy}{dx} + y^{-2} = x$$

Let $v = y^{1-n} = y^{1-3} = y^{-2}$, then the preceding equation transforms into the linear equation

$$\frac{-1}{2}\frac{dv}{dx} + v = x$$

Or

$$\frac{dv}{dx} - 2v = -2x$$

The integrating factor of this equation is $e^{\int -2dx} = e^{-2x}$.

LECTURE-6

5 Homogeneous Linear equations with constant Co-efficients:

In this section we consider the special case of the nth-order homogeneous linear differential equation in which all of the coefficients are real constants. That is, we shall be concerned with the equation

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + ...a_{n-1}\frac{dy}{dx} + a_n y = 0 \tag{29}$$

where $a_0, a_1, a_2, ..., a_n$ are real constants. We shall show that the general solution of this equation can be found explicitly. In an attempt to find solutions of a differential equation we would naturally inquire whether or not any familiar type of function might possibly have the properties that would enable it to be a solution. The differential equation (29) requires a function having the property such that if it and its various derivatives are each multiplied by certain constants, the a_i and the resulting products,$a_i f^{n-i}$, are then added, the result will equal zero for all values of x for which this result is defined. For this to be the case we need a function such that its derivatives are constant multiples of itself. Do we know of functions / having this property that

$$\frac{d^k}{dx^k}(f(x)) = cf(x)$$

for all x. The answer is 'yes', the exponential function of the form$y = e^{mx}$, where m is a constant is such that

$$\frac{d^k}{dx^k}(e^{mx}) = m^k e^{mx}$$

Thus we shall seek solutions of (29) of the form $y = e^{mx}$, where the constant m will be chosen such that e^{mx} does satisfy the equation. Assuming then that $y = e^{mx}$ is a solution for certain m, we have:

$$\begin{aligned} \frac{dy}{dx} &= me^{mx} \\ \frac{d^2y}{dx} &= m^2 e^{mx} \\ &. \\ &. \\ &. \\ \frac{d^ny}{dx^n} &= m^n e^{mx} \end{aligned}$$

Substituting these in (29), we get

$$e^{mx}\left(a_0 m^n + a_1 m^{n-1} + ... + a_{n-1} m + a_n = 0\right)$$

Since $e^{mx} \neq 0$, therefore we have

$$a_0 m^n + a_1 m^{n-1} + ... + a_{n-1} m + a_n = 0 \tag{30}$$

Three cases arise, according as the roots of (30) are real and distinct, real and repeated, or complex.

Case-1: Distinct real Roots

Suppose the roots of (30) are the n distinct real numbers, $m_1, m_2, m_3, ..., m_n$. Then $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, ..., e^{m_n x}$ are n distinct solutions of (29).

Note: Consider the nth-order homogeneous linear differential equation (29) with constant coefficients. If the auxiliary equation (30) has the n distinct real roots $m_1, m_2, m_3, ..., m_n$, then the general solution of (29) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$$

Where, $c_1, c_2, ..., c_n$ are arbitrary constants.

Case-2: Repeated real roots: Suppose the roots of (30) are the real numbers, $m, m, m_3, ..., m_n$, in which two roots are repeated and the remaining $n-2$ roots are distinct. Then the linearly independent solution corresponding to the repeated root m is given by $y = c_1 e^{mx} + x c_2 e^{mx}$. Therefore the solution of (29) in this case is given by

$$y = c_1 e^{mx} + x c_2 e^{mx} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Case-3: Conjugate Complex Roots: Now suppose that the auxiliary equation has the complex number$a+ib$ as a non repeated root. Then, since the coefficients are real, the conjugate complex number $a-ib$ is also a non repeated root. The corresponding part of the general solution is

$$k_1e^{(a+ib)x}+k_2e^{(a-ib)x}$$

where k_1 and k_2 are arbitrary constants. The solutions defined by $k_1e^{(a+ib)x}$, $k_21e^{(a-ib)x}$ are complex functions of the real variable x. It is desirable to replace these by two real linearly independent solutions. This can be accomplished by using Euler's formula,

$$e^{i\theta}=cos\theta+isin\theta$$

which holds for all θ. using this we have

$$\begin{aligned}k_1e^{(a+ib)x}+k_2e^{(a-ib)x} =&e^{ax}\left[k_1(cosbx+isinbx)+k_2(cosbx-isinbx)\right]\\ =&e^{ax}\left[(k_1+k_2)cosbx+(k_1-ik_2)sinbx\right]\\ =&e^{ax}\left[c_1cosbx+c_2sinbx\right]\end{aligned}$$

Thus the part of the general solution corresponding to the non repeated conjugate complex roots $a\pm b$ is

$$e^{ax}\left[c_1cosbx+c_2sinbx\right]$$

LECTURE-7

6 Symbolic Operators:

The linear equation (29) can also be written as

$$\begin{aligned}(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+...a_{n-1}D+a_n)y =&0 \text{ where } D^n=\frac{d^n}{dx^n} \qquad (31)\\ f(D)y =&0\end{aligned}$$

where $f(D) = (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + ...a_{n-1}D + a_n)$. The operator D^n is called the symbolic operator. The auxiliary equation (30) is obtained from equation(31) by replacing $D = m$. The solution obtained for the auxiliary equation (30) through the different cases discussed above is called **Complementary function (C.F)**. For the Non- Homogeneous equation, there is one more solution (particular solution) called the **Particular Integral (P.I)** which contains no arbitrary constants. The complementary Function contains as many arbitrary constants as the order of the given differential equation.
Example 6.1: Solve the differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 44y = 0$$

Solution: Writing the given differential equation in symbolic form, we have

$$(D^2 - 7D - 44)y = 0$$

The auxiliary equation is $m^2 - 7m - 44 = 0$ or $(m - 11)(m + 4) = 0$ or $m = 11, m = -4$.
Therefore, the general solution is $y = c_1e^{11x} + c_2e^{-4x}$
Example 6.2: Find the general solution of

$$\frac{d^2y}{dx} + 4y = 0$$

Solution: Writing the given differential equation in symbolic form, we have

$$(D^2 - 7D - 44)y = 0$$

The auxiliary equation is $m^2 + 4 = 0$ or $m = \pm 2i$.
Therefore, the general solution is $y = (c_1cos2x + c_2sin2x)$

Now we discuss different cases for obtaining the Particular integral of the Non-Homogeneous differential equations. That is we consider the equation of the type

$$f(D)y = X \tag{32}$$

Note 1: The operators D, D^2,... represent the differentiation while as the operators $\frac{1}{D}$, $\frac{1}{D^2}$,... represent the integration.

Case-1: When X in equation (32) is of the form e^{ax}
The general solution in this case is sum of C.F and P.I.
The Particular Integral (P.I) is obtained by

$$P.I = \frac{X}{f(D)} = \frac{e^{ax}}{f(a)} \text{ provided } f(a) \neq 0$$

Example 6.3: Solve the equation

$$\left(D^2 + 5D + 6\right) y = e^{2x}$$

Solution: The auxiliary equation is $m^2 + 5m + 6 = 0$ or $m = -2, -3$.
Therefore, the $C.F = c_1 e^{-2x} + c_2 e^{-3x}$
Now the P.I. is equal to

$$\frac{1}{(D+2)(D+3)e^{2x}} = \frac{1}{(2+2)(2+3)e^{2x}} = \frac{1}{20} e^{2x}$$

Hence, the general solution is

$$c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{20} e^{2x}$$

Case-2: When X in equation (32) is of the form $sinax$ or $cosax$, then P.I=$\frac{sinax \text{ or } cosax}{f(-a^2)}$ provided $f(-a^2) \neq 0$

Example 6.4: Solve the equation

$$\left(D^2 + D + 1\right) y = sin2x$$

Solution: The auxiliary equation is $m^2 + m + 1 = 0$ or $m = \frac{-1 \pm \sqrt{3}}{2}$.
Therefore, the C.F. is

$$e^{-\frac{x}{2}}(c_1 cos\frac{\sqrt{3}}{2} + c_2 sin\frac{\sqrt{3}}{2})$$

Now the P.I. is equal to

$$\frac{sin2x}{D^2 + D + 1} = \frac{sin2x}{-4 + D + 1} = \frac{sin2x}{D - 3} = \frac{(D+3)sin2x}{D^2 - 9}$$

Thus, P.I. is

$$\frac{D(sin2x)+3sin2x}{-4-9}=\frac{2cos2x}{-13}$$

Hence the general solution is

$$e^{-\frac{x}{2}}(c_1cos\frac{\sqrt{3}}{2}+c_2sin\frac{\sqrt{3}}{2})-\frac{2cos2x}{-13}$$

Case-3: When X in equation (32) is of the form x^m, then P.I=$\frac{x^m}{f(D)}$
In this case, $f(D)$ is evaluated with the help of Binomial theorem after taking out the lowest degree term from $f(D)$.
Example 6.5: Solve the equation $(2D^2+5D+2)\,y=5+2x$
Solution: The auxiliary equation is $2m^2+5m+2=0$ or $m=-2,\frac{-1}{2}$.
The C.F is

$$c_1e^{-2x}+c_2e^{-\frac{x}{2}}$$

The P.I is

$$\begin{aligned}\frac{5+2x}{2D^2+5D+2}=&\frac{1}{2}\left(1+\frac{5D}{2}+...\right)^{-1}(5+2x)\\ \text{Or,}\;\; P.I=&\frac{1}{2}\left(1+\frac{5D}{2}+...\right)(5+2x)\\ =&\frac{1}{2}\left(5+2x-\frac{5}{2}.2\right)=x\end{aligned}$$

Hence the general solution is $c_1e^{-2x}+c_2e^{-\frac{x}{2}}+x$
Case-4: When X in equation (32) is of the form $e^{ax}V$, where V is any function of x then P.I=$e^{ax}\frac{1}{f(D+a)}V$
Example 6.6: Solve the equation $(D^2-4D+3)\,y=e^{2x}sin3x$
Solution: The auxiliary equation is $m^2-4m+3=0$ or $m=1,3$.
The C.F is $c_1e^x+c_2e^{3x}$
The P.I is

$$\frac{1}{(D-1)(D-3)}e^{2x}sin3x=e^{2x}$$
$$\frac{1}{(D+2-1)(D+2-3)}sin3x=\frac{1}{(D^2-1)}sin3x=\quad e^{2x}\frac{1}{-9-1}sin3x$$

Thus P.I= $e^{2x}\frac{1}{-10}sin3x$
The general solution is $c_1e^x + c_2e^{3x} + e^{2x}\frac{1}{-10}sin3x$
Case-5: When X in equation (32) is of the form e^{ax} and $f(a) = 0$
In this case $(D - a)$ is a factor of $f(D)$. Therefore, we can write $f(D) = (D - a)\phi(D)$, where $\phi(a) \neq 0$.
Therefore, the P.I is given by

$$\begin{aligned} P.I =& \frac{e^{ax}}{(D-a)\phi(D)} \\ =& \frac{1}{\phi(a)}e^{ax}\left(\frac{1}{D+a-a}\right)1 \\ =& \frac{x.e^{ax}}{\phi(a)} \end{aligned}$$

Example 6.7: Solve the equation $(D^2 - D - 6)\,y = e^{-2x}$
Solution: The auxiliary equation is $m^2 - m - 6 = 0$ or $m = 3, -2$.
The C.F=$c_1e^{3x} + c_2e^{-x}$
P.I= is equal to

$$\frac{1}{(D-3)(D+2)}e^{-2x} = \frac{1}{(-2-3)(D+2)}e^{-2x} = \frac{1}{-5}xe^{-2x}$$

Example 6.8: Solve the differential equation

$$(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$$

Solution: The auxiliary equation is $m^3 + 2m^2 + m = 0$. The roots are $m = -1, -1, 0$. Therefore, the C.F=$c_1 + (c_2 + xc_3)e^{-x}$
Now, P.I. for e^{2x} is

$$\frac{e^{2x}}{D(D+1)^2} = \frac{1}{18}e^{2x}$$

P.I. for $(x^2 + x) =$

$$\begin{aligned} \frac{1}{D(D+1)^2}(x^2+x) =& \frac{1}{D}(D+2)^{-2}(x^2+x) \\ =& \frac{1}{D}(1 - 2D + 3D^2 ...)(x^2+x) \\ =& \frac{1}{D}(x^2 + x - 4x - 2 + 6) = \frac{x^3}{3} - \frac{3}{2}x^2 + 4x \end{aligned}$$

Thus the general solution is

$$c_1 + (c_2 + xc_3)e^{-x} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x$$

Example 6.9: Solve the differential equation

$$(D^2 + D - 2)y = x + sinx$$

Solution: The auxiliary equation is $m^2+m-2 = 0$. The roots are $m = -2, 1$. Therefore, the C.F is equal to

$$c_1e^{-2x} + c_2e^x$$

Now, P.I for x is

$$\begin{aligned}\frac{x}{D^2 + D - 2} &= \frac{-1}{2}\left(1 - \frac{D}{2}\right)^2 x\\ &= \frac{-1}{2}\left(1 + \frac{D}{2} + ...\right) x\\ &= \frac{-1}{4}(2x + 1)\end{aligned}$$

P.I. for $sinx$ is

$$\begin{aligned}\frac{sinx}{D^2 + D - 2} &= \frac{sinx}{-1 + D - 2} = \qquad \frac{(D + 3)sinx}{D^2 - 9}\\ &= \frac{D(sinx) + 3sinx}{-1 - 9}\\ &= \frac{cosx + 3sinx}{-10}\end{aligned}$$

The general solution is

$$c_1e^{-2x} + c_2e^x + \frac{cosx + 3sinx}{-10}$$

Case-6: When X in equation (32) is of the form $sinax$ or $cosax$ and $f(D) = 0$ when $D^2 = -a^2$

In this case

$$\frac{siax \text{ or} cosax}{D^2 + a^2}$$

will not be evaluated by putting $D^2 = -a^2$. In such cases we shall calculate P.I for $e^{iax} = cosax + isinax$.
Therefore, real part of P.I for e^{iax}= P.I. for cos ax and
real part of P.I for e^{iax}= P.I. for cos ax
Now,

$$\begin{aligned}\frac{e^{iax}}{D^2+a^2} &= \frac{e^{iax}}{(D+ia)(D-ia)} \\ &= \frac{e^{iax}}{(D+ia)(D-ia)} = \frac{e^{iax}}{(ia+ia)(D-ia)} \\ &= \frac{e^{iax}}{2ia}\left(\frac{1}{D+ia-ia}.1\right) \\ &= \frac{e^{iax}}{2ia.\frac{1}{D}}.1 = \frac{x}{2ia}(e^{iax}) \\ &= -\frac{ix}{2a}cosax + \frac{x}{2a}sinax\end{aligned}$$

Thus Particular Integral of

$$\frac{cosax}{D^2+a^2} = \frac{x}{2a}sinax$$

and of

$$\frac{sinax}{D^2+a^2} = -\frac{x}{2a}cosax$$

Example 6.10: Solve the Differential equation

$$(D^2+4)y = cos2x + sin2x$$

Solution: If we replace D^2 by -2^2, then D^2+4 vanishes. hence the P.I. is to be obtained by the above procedure discussed in Case-6.

$$\begin{aligned}\frac{e^{i2x}}{D^2+4} &= \frac{e^{i2x}}{(D+2i)(D-2i)}\\ &= \frac{e^{i2x}}{(ia+ia)(D-ia)}\\ &= \frac{e^{i2x}}{4i}\left(\frac{1}{D+2i-2i}.1\right)\\ &= \frac{e^{i2x}}{4i.\frac{1}{D}}.1 = \frac{x}{4i}(e^{i2x})\\ &= -\frac{ix}{4}cos2x + \frac{x}{4}sin2x\end{aligned}$$

Hence the P.I. of

$$\frac{cos2x + sin2x}{D^2+4}$$

is

$$\frac{x}{4}sin2x - \frac{x}{4}cos2x$$

Example 6.11: Solve the differential equation

$$\frac{d^2y}{dx^2} + 4y = xsinx$$

Solution: Here,the auxiliary equation is $m^2 + 4 = 0$ implies $m = -2i, 2i$.
Therefore, the C.F. is $c_1cos2x + c_2sin2x$
Now, P.I. for $xsinx$ is

$$\frac{xsinx}{D^2+4} = \text{ imaginary part of } \frac{xe^{ix}}{D^2+4}$$

$$=\text{Imaginary part of } e^{ix}\left(\frac{1}{(D+i)^2+4}\right)x$$
$$=\text{Imaginary part of } e^{ix}\left(\frac{1}{D^2+2Di+3}\right)x$$
$$=\text{Imaginary part of } \frac{e^{ix}}{3}\left(1+\frac{2Di}{3}...\right)^{-1}x$$
$$=\text{Imaginary part of } \frac{e^{ix}}{3}\left(1-\frac{2Di}{3}...\right)x$$
$$=\text{Imaginary part of } \frac{e^{ix}}{3}\left(1-\frac{2}{3}i...\right)x$$
$$=\text{Imaginary part of } \frac{e^{ix}}{3}\left(x-\frac{2}{3}i\right)$$
$$=\text{Imaginary part of } \frac{(cosx+isinx)(3x-2i)}{9}$$
$$=\frac{3xsinx-2cosx}{9}$$

Hence the general solution is

$$c_1cos2x+c_2sin2x+\frac{3xsinx-2cosx}{9}$$

Exercises for the reader: Solve the following differential equations:
1. $(D^3+4D^2+4D)y=8e^{-2x}+x^2$
2. $(D^3+1)y=8+e^{-x}+5e^{-2x}$
3. $(D^2-2D+5)y=e^{2x}sinx$

LECTURE-8

7 Equations reducible to homogeneous equations with constant coefficients

The homogeneous linear equation of the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+...a_{n-1}x\frac{dy}{dx}+a_ny=X$$

where $a_1, a_2, ..., a_n$ are real constants and X is a function of x.
By a little substitution, we show that the above linear equation can be transformed into linear equation with constant coefficients.
Let $x = e^z$ so that $z = logx$ and $\frac{dz}{dx} = \frac{1}{x}$
Now,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}\\ \frac{d^2y}{dx^2} &= \frac{x\frac{d^2y}{dz^2}.\frac{dz}{dx} - \frac{dy}{dz}}{x^2}\\ \text{Or, } x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz}\end{aligned}$$

If we put $\frac{d}{dz} = D$, then we have from above

$$\begin{aligned}x\frac{dy}{dx} &= Dy\\ x^2\frac{d^2y}{dx^2} &= D(D-1)y\\ &.\\ &.\\ &.\\ x^n\frac{d^ny}{dx^n} &= D(D-1)(D-2)...(D-n+1)y\end{aligned}$$

Example 7.1: Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$$

Solution: Let $x = e^z$ so that $z = logx$ and $\frac{dz}{dx} = \frac{1}{x}$
Then,

$$\begin{aligned}\left[D(D-1) - 3D + 4\right]y &= 2e^{2z}\\ \left[D^2 - 4D + 4\right]y = 2e^{2z}\end{aligned}$$

The auxiliary equation is $(m-2)^2 = 0$ or $m = 2, 2$.
Therefore, C.F=$(c_1 + zc_2)e^{2z}$

Now, Particular integral is

$$\begin{aligned} P.I =& \frac{2e^{2z}}{(D-2)^2} \\ =& 2e^{2z}.\frac{1}{(D+2-2)^2}.1 \\ =& 2e^{2z}.\frac{1}{D^2}.1 = 2e^{2z}.\frac{z^2}{2} = z^2e^{2z} \end{aligned}$$

Hence the general solution is $y = (c_1 + zc_2)e^{2z} + z^2e^{2z}$. Now substituting back $z = logx$ and $e^x = x$, we have $y = (c_1 + c_2logx)x^2 + x^2(logx)^2$

Example 7.2: Solve the differential equation

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{1}{x}$$

Solution: Let $x = e^z$ so that $z = logx$ and $\frac{dz}{dx} = \frac{1}{x}$ and let $D = \frac{d}{dz}$

Then,

$$\begin{aligned} [D(D-1)(D-2) + 2D(D-1) - D + 1]\, y =& \frac{1}{e^z} \\ \text{Or,} \quad \left[D^3 - D^2 - D + 1\right] y =& e^{-z} \end{aligned}$$

The auxiliary equation is $(m-1)^2(m+1) = 0$ or $m = 1, 1, -1$.

Therefore, C.F=$(c_1 + zc_2)e^z + c_3e^{-z}$

Now, Particular integral is

$$P.I = \frac{e^{-z}}{(D-1)^2(D+1)} = \frac{e^{-z}.1}{(-1-1)^2(D+1)} = \frac{e^{-z}}{4(D-1+1)}.1 = \frac{ze^{-z}}{4}$$

Hence the general solution is $y = (c_1 + zc_2)e^z + c_3e^{-z} + \frac{ze^{-z}}{4}$

Thus, $y = (c_1 + c_logx)x + \frac{c_3}{x} + \frac{1}{4x}(logx)$

Exercises for the reader: Solve the following differential equations

1. $x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^2 + 3x$
2. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$
3. $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = xlogx$
4. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2sin(logx)$

Differential equations, Integration of irrational functions & Reduction formulae

Lecture Notes

Mathematics *S.P. College*

1.1 Differential equations

Bernoulli's equation

Any differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{1.1}$$

where P and Q are either constants or functions of x alone.

To solve such differential equation, we divide both sides of (1.1) by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q. \tag{1.2}$$

Put $y^{1-n} = z$ then

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}.$$

For (1.2), we get

$$\begin{aligned} \frac{1}{1-n}\frac{dz}{dx} + Pz &= Q \\ \frac{dz}{dx} + P(1-n)z &= (1-z)Q \\ \frac{dz}{dx} + P'z &= Q', \end{aligned} \tag{1.3}$$

where $P' = P(1-n)$ and $Q' = (1-z)Q$, is a linear differential equation.

Therefore, $I.F = e^{\int P'dx}$, multiplying both sides of (1.3) by $e^{\int P'dx}$, we get

$$\frac{d}{dx}\left(e^{\int P'dx}z\right) = Q'e^{\int P'dx}.$$

Integrate both sides, we get

$$e^{\int P'dx}z = \int Q'e^{\int P'dx}dx + c$$

gives the solution of given differential equation.

Example

Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = y^2. \tag{1.4}$$

Solution. Divide both sides of (1.4) by y^2, we get

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1.$$

Put $y^{-1} = t$, then we have

$$\begin{aligned} (-1)y^{-2}\frac{dy}{dx} &= \frac{dt}{dx} \\ y^{-2}\frac{dy}{dx} &= -\frac{dt}{dx}. \end{aligned}$$

From above equation, we get

$$\begin{aligned} -\frac{dt}{dx} + \frac{1}{x}t &= 1 \\ \frac{dt}{dx} - \frac{1}{x}t &= -1. \end{aligned} \tag{1.5}$$

It is a linear differential.

Therefore $I.F = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = 1/x$. Multiplying both sides of (1.5) by $1/x$, we get

$$\frac{d}{dx}\left(\frac{1}{x}t\right) = -\frac{1}{x}.$$

Integrating both sides, we get

$$\begin{aligned} \frac{t}{x} &= -\log x + c \\ \text{or}\quad t &= -x\log x + cx \\ y^{-1} &= -x\log x + cx \\ \frac{1}{y} &= x(-\log x + c) \qquad \blacksquare \end{aligned}$$

Exact differential equation

The equation $Mdx + Ndy = 0$, where M & N are functions of x and y, is said to be an exact differential equation if $Mdx + Ndy = 0$ is the exact differential function of x and y i.e., $Mdx + Ndy = du$, where u is a function of x & y.

Art: The necessary and sufficient condition for the equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof. If $Mdx + Ndy = 0$ is exact, therefore $Mdx + Ndy = du$ where $u = f(x, y)$.

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow M = \frac{\partial u}{\partial x}, \quad \rightarrow \quad (i) \qquad N = \frac{\partial u}{\partial y} \quad \rightarrow \quad (ii).$$

Differential both sides of (i) partially with respect to y and (ii) partially with respect to x, we get

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Conversely, suppose

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Let $s = \int Mdx$ therefore $M = \partial s / \partial x$ and

$$\frac{\partial M}{\partial y} = \frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y}$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial s}{\partial y}\right).$$

Integrating both sides, we get

$$N = \frac{\partial s}{\partial y} + F(y), \quad \text{where } F(y) \text{ is a function of } y \text{ alone.}$$

Now,

$$\begin{aligned} Mdx + Ndy &= \frac{\partial s}{\partial x}dx + \left(\frac{\partial s}{\partial y} + F(y)\right)dy \\ &= \left(\frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy\right) + F(y)dy \\ &= dx + d\left(\int F(y)dy\right) \\ &= d\left(s + \int F(y)dy\right) = d\varphi, \quad \text{where} \quad \varphi = s + \int F(y)dy. \end{aligned}$$

$\Rightarrow \quad Mdx + Ndy$ is exact.
To obtain its solution, we proceed as follows:
Since $Mdx + Ndy = 0 \quad \Rightarrow \quad d\varphi = 0 \quad \Rightarrow \quad \varphi =$ constant. This implies

$$\int Mdx + \int F(y)dy = \text{constant} \quad \blacksquare$$

Example

Solve the differential equation

$$\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-x/y\right)dy=0.$$

Solution. Comparing this differential equation with $Mdx+Ndy=0$, we get

$$M=1+e^{x/y};\quad N=e^{x/y}\left(1-x/y\right).$$

Therefore,

$$\frac{\partial M}{\partial y}=e^{x/y}\left(-\frac{x}{y^2}\right)$$

and

$$\begin{aligned}\frac{\partial N}{\partial x}&=e^{x/y}.\frac{1}{y}\left(1-\frac{x}{y}\right)+e^{x/y}\left(-\frac{1}{y}\right)\\&=e^{x/y}\frac{1}{y}-\frac{x}{y^2}e^{x/y}-e^{x/y}\frac{1}{y}=-e^{x/y}\frac{x}{y^2}.\end{aligned}$$

Therefore $\partial M/\partial y=\partial N/\partial x$, i.e., this differential equation is exact. Its solution is obtained as

$$\int Mdx+\int(\text{terms in } N \text{ independent of } x)dy=k$$

$$\Rightarrow\int\left(1+e^{x/y}\right)dx+\int 0.dx=k$$

$$x+\frac{e^{x/y}}{1/y}=k,\quad\text{or}\quad x+ye^{x/y}=k\quad\blacksquare$$

Linear Differential equations (LDE) with constant coefficients

Operator: The part d/dx of the symbol dy/dx may be regarded as an operator, such that when it operates on y, the result is derivative of y with respect to x.

Similarly $d^2/dx^2, d^3/dx^3, \ldots, d^n/dx^n$ may be regarded as operator. In symbolic form, we have

$$\frac{d}{dx}=D;\quad\frac{d^2}{dx^2}=D^2;\quad\frac{d^3}{dx^3}=D^3;\quad\ldots\quad;\frac{d^n}{dx^n}=D^n.$$

LDE with constant coefficients

A linear differential equation with constant coefficients is that in which the dependent variable and its differential coefficients occur only in the first degree and are not multiples together and the coefficients are all constants.

Therefore, the equation of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_ny=X,\tag{1.6}$$

where $a_0, a_1, \ldots, a_n$ are all constants and X is a function of x, is called LDE with constant coefficients.

The above equation can also be written as

$$\begin{aligned} & a_0 D^n y + a_1 D^{n-1} y + \ldots + a_n y = X, \qquad \text{where} \quad D = \frac{d}{dx} \\ \text{or} \quad & \left(a_0 D^n + a_1 D^{n-1} + \ldots + a_n\right) y = X. \end{aligned}$$

Here $a_0 D^n + a_1 D^{n-1} + \ldots + a_n$ is a function of operator D. Thus $a_0 D^n + a_1 D^{n-1} + \ldots + a_n = f(D)$ is regarded as a single operator operating on y.
NOTE 1 If $y = y_1, y = y_2, \ldots, y = y_n$ are n linearly independent solutions of

$$\left(a_0 D^n + a_1 D^{n-1} + \ldots + a_n\right) y = 0 \tag{1.7}$$

then, $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is the general solution or complete solution of (1.7) where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Since given differential equation is

$$a_0 D^n y + a_1 D^{n-1} y + \ldots + a_n y = 0 \tag{1.8}$$

has $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ as its solutions. This implies

$$\begin{aligned} a_0 D^n y_1 + a_1 D^{n-1} y_1 + \ldots + a_n y_1 &= 0 \\ a_0 D^n y_2 + a_1 D^{n-1} y_2 + \ldots + a_n y_2 &= 0 \\ &\vdots \\ a_0 D^n y_n + a_1 D^{n-1} y_n + \ldots + a_n y_n &= 0 \end{aligned} \tag{1.9}$$

Substitute $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ in right hand side of (1.8), we have

$$\begin{aligned} & a_0 D^n \left(c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\right) \\ & \quad + a_1 D^{n-1} \left(c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\right) \\ & \quad + \ldots + a_n \left(c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\right) \\ = & c_1 \left(a_0 D^n y_1 + a_1 D^{n-1} y_1 + \ldots + a_n y_1\right) \\ & \quad + c_2 \left(a_0 D^n y_2 + a_1 D^{n-1} y_2 + \ldots + a_n y_2\right) \\ & \quad + \ldots + \left(a_0 D^n y_n + a_1 D^{n-1} y_n + \ldots + a_n y_n\right) \\ = & c_1(0) + c_2(0) + \ldots + c_n(0) = 0 \qquad (\text{by } (1.9)). \end{aligned}$$

Thus equation (1.8) is satisfied by

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Hence it is the complete solution of given differential equation.
NOTE 2: *Auxiliary equation*
Consider the differential equation

$$\left(a_0 D^n + a_1 D^{n-1} + \ldots + a_n\right) y = 0.$$

Let $y = e^{mx}$ be its solution, then

$$Dy = De^{mx} = me^{mx}$$

$$
\begin{aligned}
D^2y &= D^2e^{mx} = m^2e^{mx} \\
D^3y &= D^3e^{mx} = m^3e^{mx} \\
&\vdots \\
D^ny &= D^ne^{mx} = m^ne^{mx}
\end{aligned}
$$

Substituting these values in (1.8), we get

$$
\begin{aligned}
a_0m^ne^{mx} + a_1m^{n-1}e^{mx} + \ldots + a_ne^{mx} &= 0 \\
\left(a_0m^n + a_1m^{n-1} + \ldots + a_n\right)e^{mx} &= 0.
\end{aligned}
$$

Since $e^{mx} \neq 0,\ \forall\ m, x$. Therefore

$$a_0m^n + a_1m^{n-1} + \ldots + a_n = 0. \tag{1.10}$$

Hence e^{mx} is a solution of (1.8) if m satisfies (1.10). Therefore equation (1.10) is called the *Auxiliary equation* for the differential equation (1.8).

NOTE 3:

Case I If all the roots of an auxiliary equation are real and distinct i.e., if $m = m_1, m_2, \ldots, m_n$ are roots of auxiliary equation and are real & distinct, then

$$y = e^{m_1x}; \quad y = e^{m_2x}; \quad \ldots \quad , y = e^{m_nx}$$

are n-independent solutions of (1.8). Therefore the complete solution of (1.8) is given by

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}.$$

Case II When two roots of an auxiliary equation are equal and all others different say $m_1 = m_2$ then the roots of auxiliary equation are $m_1, m_2, \ldots, m_n$.

The complete solution in this case is given by

$$
\begin{aligned}
y &= c_1e^{m_1x} + xc_2e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\
\text{or}\quad y &= (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}.
\end{aligned}
$$

Case III When two roots of an auxiliary equation are imaginary and conjugate to each other and rest are real and different, say $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ then the complete solution is given by

$$
\begin{aligned}
y &= c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\
&= c_1e^{\alpha x}e^{i\beta x} + c_2e^{\alpha x}e^{-i\beta x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\
&= e^{\alpha x}\left(c_1e^{i\beta x} + c_2e^{-i\beta x}\right) + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\
&= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] \\
&\qquad + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\
&= e^{\alpha}\left[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\right] \\
&\qquad + c_3e^{m_3x} + \ldots + c_ne^{m_nx}.
\end{aligned}
$$

Thus

$$y = e^{\alpha}\left[A\cos\beta x + B\sin\beta x\right] + c_3e^{m_3x} + \ldots + c_ne^{m_nx},$$

where $A = (c_1 + c_2)$ & $B = i(c_1 - c_2)$, is the complete solution.

NOTE 4: If in a linear differential equation

$$a_0D^ny + a_1D^{n-1}y + \ldots + a_ny = X$$

(i) $X = 0$ then the complete solution is given by

$$y = \text{complementary function=C.F}$$

(ii) $X \neq 0$ then the complete solution is given by

$$y = \text{C.F+P.I} \quad \text{where P.I= particular integral.}$$

Examples

(1) Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$.

Solution. The given equation in symbolic form is

$$D^2y - 3Dy - 4y = 0$$
$$\left(D^2 - 3D - 4\right) y = 0$$

The auxiliary equation is

$$m^2 - 3m - 4 = 0$$
$$m^2 - 4m + m - 4 = 0$$
$$m(m-4) + (m-4) = 0$$
$$(m-4)(m+1) = 0.$$

This gives $m = 4$ or $m = -1$, therefore roots of auxiliary equation are real and distinct. Hence complete solution is given differential equation is given by
Complete solution = C.F i.e.,

$$y = c_1e^{4x} + c_2e^{-x} \qquad \blacksquare$$

(2) Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$.

Solution. The equation in symbolic form can be written as

$$D^2y - 2Dy + y = 0$$
$$(D^2 - 2D + 1)y = 0.$$

It's auxiliary equation is

$$m^2 - 2m + 1 = 0$$

this gives $m = 1, 1$. Hence roots are identical or (repeated) then the complete solution is

$$y = (c_1 + xc_2)e^x \qquad \blacksquare$$

(3) Solve $(D^3 - D^2 - D - 2)y = 0$.

Solution. Its auxiliary equation is

$$m^3 - m^2 - m - 2 = 0.$$

By inspection one can easily verify $m = 2$ satisfies it. Therefore by synthetic division

$$\begin{array}{r|rrrr} m=2 & 1 & -1 & -1 & -2 \\ \hline & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

This, auxiliary equation is equivalent to

$$(m-2)(m^2+m+1)=0.$$

This gives $m = 2, \dfrac{-1 \pm i\sqrt{3}}{2}$.

Thus the complete solution is

$$\begin{aligned} y &= c_1 e^{2x} + c_2 e^{\left(\frac{-1+i\sqrt{3}}{2}\right)x} + c_3 e^{\left(\frac{-1-i\sqrt{3}}{2}\right)x} \\ &= c_1 e^{2x} + e^{-\frac{x}{2}} \left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right) \end{aligned} \quad \blacksquare$$

Exercise

Solve $\left(D^4 + a^4\right) y = 0$ ■

As we mentioned earlier the differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = X, \quad \text{where} \quad X \neq 0$$

has complete solution of the form, complete solution= C.F+P.I.

For particular integral of various differential equation we proceed as,

$$\frac{1}{D} \text{ represents integral operator.}$$

Therefore $\frac{1}{D}\sin x = \int \sin x dx = -\cos x$; $\frac{1}{D}e^{mx} = \int e^{mx}dx = \frac{e^{mx}}{m}$.
NOTE 5:

$$\begin{aligned} De^{ax} &= ae^{ax} \\ D^2 e^{ax} &= D\left(De^{ax}\right) = D(ae^{ax}) = aD(e^{ax}) = a^2 e^{ax}. \end{aligned}$$

Similarly

$$D^3 e^{ax} = a^2 e^{ax}; \quad D^4 e^{ax} = a^4 e^{ax}; \quad \ldots \quad ; D^n e^{ax} = a^n e^{ax}.$$

In general if $f(D)$ is a polynomial in D then $f(D)e^{ax} = f(a)e^{ax}$. This implies

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \quad \text{provided} \quad f(a) \neq 0.$$

In case $f(a) = 0$ we proceed as follows:
Case I

$$\frac{1}{D-a}e^{ax} = y \quad \text{say}$$

operating both sides by $(D-a)$, we get

$$e^{ax} = (D-a)y$$
$$\left(\frac{d}{dx} - a\right) y = e^{ax}$$
$$\frac{dy}{dx} - ay = e^{ax} \tag{1.11}$$

This is linear differential equation of the form $dy/dx + Py = Q$. Here $P = -a$ and $Q = e^{ax}$.

Integrating factor= $e^{\int P dx} = e^{\int -a dx} = e^{-ax}$.
Multiplying both sides of (1.11) by e^{-ax}, we get

$$e^{-ax}\left(\frac{dy}{dx} - ay\right) = e^{ax}e^{-ax}$$
$$\frac{d}{dx}\left(ye^{-ax}\right) = 1$$

integrating both sides, we get

$$ye^{-ax} = x$$
$$y = xe^{ax}$$

i.e.,

$$\frac{1}{D-a}e^{ax} = xe^{ax}.$$

Case II

$$\frac{1}{(D-a)^2}e^{ax} = \frac{1}{D-a}\left(\frac{1}{D-a}e^{ax}\right)$$
$$= \frac{1}{D-a}\left(xe^{ax}\right).$$

$$\text{Let} \quad \frac{1}{D-a}\left(xe^{ax}\right) = y \quad \text{(say)}$$

operating both sides by $(D-a)$, we get

$$xe^{ax} = (D-a)y$$
$$\left(\frac{d}{dy} - a\right) y = xe^{ax}$$
$$\frac{dy}{dx} - ay = xe^{ax}. \tag{1.12}$$

It is a linear differential equation and here integrating factor= $e^{\int -a dx} = e^{-ax}$. Multiplying both sides of (1.12) by e^{-ax}, we get

$$e^{-ax}\left(\frac{dy}{dx} - ay\right) = e^{ax}(xe^{-ax})$$

$$\frac{d}{dx}\left(ye^{-ax}\right) = x$$

integrating both sides, we get

$$ye^{-ax} = \frac{x^2}{2}$$
$$y = \frac{x^2}{2}e^{ax}$$

i.e.,

$$\frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2}e^{ax}.$$

Similarly, one can prove

$$\frac{1}{(D-a)^3}e^{ax} = \frac{x^3}{3!}e^{ax}$$
$$\frac{1}{(D-a)^4}e^{ax} = \frac{x^4}{4!}e^{ax}$$
$$\vdots$$
$$\frac{1}{(D-a)^n}e^{ax} = \frac{x^n}{n!}e^{ax}.$$

Rule II: $\frac{1}{f(D)}e^{ax}$ where $f(a) = 0$. In such a case $(D-a)$ is a factor of $f(D)$. The above expression can also be written as

$$\frac{1}{(D-a)g(D)}e^{ax}$$

where $g(D)$ is a polynomial in D of degree one less than that of $f(D)$.

Examples

(1)

$$\begin{aligned}\frac{1}{D^2-1}e^x &= \frac{1}{(D+1)(D-1)}e^x \\ &= \frac{1}{D-1}\frac{1}{1+1}e^x \\ &= \frac{1}{2}\frac{1}{D-1}e^x = \frac{1}{2}xe^x \qquad \blacksquare\end{aligned}$$

(2)

$$\begin{aligned}\frac{1}{D^3-3D^2+3D-1}e^x &= \frac{1}{(D-1)^3}e^x \\ &= \frac{x^3}{3!}e^x = \frac{x^3}{6} \qquad \blacksquare\end{aligned}$$

Rule III

$$D\sin ax = a\cos ax$$

$$D^2 \sin ax = D(D\sin ax) = D(a\cos ax) = -a^2 \sin ax$$
$$D^3 \sin ax = D(D^2 \sin ax) = D(-a^2 \sin ax) = -a^3 \cos ax$$
$$D^4 \sin ax = D(D^3 \sin ax) = D(-a^3 \cos ax) = (-a^2)^2 \sin ax.$$

In general,

$$D^{2n} \sin ax = (-a^2)^n \sin ax.$$

Therefore,

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax \quad \text{provided} \quad f(-a^2) \neq 0.$$

In case $f(-a^2) = 0$, we may proceed as in the following example.

$$\begin{aligned}
\frac{1}{D^2 + a^2} \sin ax &= \frac{1}{(D+ia)(D-ia)} \left(\frac{e^{iax} - e^{-iax}}{2i} \right) \quad (\text{here} \quad f(-a^2) = 0) \\
&= \frac{1}{2i} \left[\frac{1}{(D+ia)(D-ia)} e^{iax} - \frac{1}{(D+ia)(D-ia)} e^{-iax} \right] \\
&= \frac{1}{2i} \left[\frac{1}{2ia} \frac{1}{D-ia} e^{iax} - \frac{1}{(-2ia)} \frac{1}{(D+ia)} e^{-iax} \right] \\
&= -\frac{1}{4a} \left[xe^{iax} + xe^{-iax} \right] \\
&= -\frac{x}{2a} \left(\frac{e^{ax} + e^{-iax}}{2} \right) = -\frac{x}{2a} \cos ax.
\end{aligned}$$

Similarly

$$\frac{1}{D^2 - a^2} \cos ax = \frac{x}{2a} \sin ax.$$

Rule IV

$$\frac{1}{(D^2 - a^2)^n} \sin ax = \text{Im} \left(\frac{1}{D^2 - a^2} e^{iax} \right)$$

$$\frac{1}{(D^2 - a^2)^n} \cos ax = \text{Re} \left(\frac{1}{D^2 - a^2} e^{iax} \right)$$

Example

Solve the differential equation

$$(D^3 + 1)y = 3 + 5e^x.$$

Solution. The auxiliary equation is

$$\begin{aligned}
m^3 + 1 &= 0 \\
(m+1)(m^2 + m + 1) &= 0.
\end{aligned}$$

This implies $m = -1$ or $m = \frac{-1 \pm i\sqrt{3}}{2}$. Therefore complementary function is

$$C.F = c_1 e^{-x} + c_2 e^{\left(\frac{-1+i\sqrt{3}}{2}\right)x} + c_3 e^{\left(\frac{-1-i\sqrt{3}}{2}\right)x}$$

$$= c_1 e^{-x} + e^{-\frac{x}{2}} \left(c_1 e^{i\frac{i\sqrt{3}}{2}x} + c_2 e^{-\frac{i\sqrt{3}}{2}x} \right)$$
$$= c_1 e^{-x} + e^{-\frac{x}{2}} \left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right).$$

Next, the particular integral is given as

$$\begin{aligned} P.I &= \frac{1}{D^3+1}\left(3+5e^x\right) \\ &= \frac{1}{D^3+1}(3) + 5\frac{1}{D^3+1}e^x \\ &= 3\frac{1}{D^3+1}e^{0.x} + 5\frac{1}{D^3+1}e^x \\ &= 3\frac{1}{0^3+1}(1) + 5\left(\frac{1}{2}\right)e^x = 3 + \frac{5}{2}e^x. \end{aligned}$$

Thus complete solution is

$$y = c_1 e^{-x} + e^{-\frac{x}{2}} \left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right) + 3 + \frac{5}{2}e^x.$$

Rule V Evaluate

$$\frac{1}{f(D)}e^{ax}X \qquad \text{where } X \text{ is a function f } x.$$

Since if V is a function of x, therefore by successive differentiation, we have

$$\begin{aligned} D(e^{ax}V) &= e^{ax}DV + ae^{ax}V \\ &= e^{ax}(D+a)V. \end{aligned}$$

Next,

$$\begin{aligned} D^2(e^{ax}V) &= D(D(e^{ax}V)) \\ &= D(e^{ax}(D+a)V)\& = e^{ax}D(D+a)V + ae^{ax}(D+a)V \\ &= e^{ax}(D+a)[D+a]V \\ &= e^{ax}(D+a)^2V. \end{aligned}$$

Similarly,

$$\begin{aligned} D^3(e^{ax}V) &= e^{ax}(D+a)^3V \\ &\vdots \\ f(D)e^{ax}V &= e^{ax}f(D+a)V. \end{aligned}$$

Now if $f(D+a)V = X$ then

$$\frac{1}{f(D+a)}X = V.$$

Therefore

$$\begin{aligned} f(D)\left[e^{ax}\frac{1}{f(D+a)}X\right] &= e^{ax}X \\ \text{or} \quad e^{ax}\frac{1}{f(D+a)}X &= \frac{1}{f(D)}e^{ax}X \\ \text{or} \quad \frac{1}{f(D)}e^{ax}X &= e^{ax}\frac{1}{f(D+a)}X \quad \blacksquare \end{aligned}$$

Example

(1) Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x.$$

Solution. The equation can be written as

$$(D^2 - 2D + 4)y = e^x \cos x.$$

The auxiliary equation is

$$m^2 - 2m + 4 = 0.$$

This implies $m = 1 \pm i\sqrt{3}$, therefore complementary function is

$$\begin{aligned} C.F &= c_1 e^{(1+i\sqrt{3})x} + c_2 e^{(1-i\sqrt{3})x} \\ &= e^x \left(A\cos\sqrt{3}x + B\sin\sqrt{3}x \right). \end{aligned}$$

The particular integral is given by

$$\begin{aligned} P.I &= \frac{1}{D^2 - 2D + 4} e^x \cos x \\ &= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x \\ &= e^x \frac{1}{D^2 + 3} \cos x \\ &= e^x \frac{1}{(-1)^2 + 3} \cos x = \frac{1}{2} e^x \cos x. \end{aligned}$$

Thus complete solution is

$$y = e^x \left(A\cos\sqrt{3}x + B\sin\sqrt{3}x \right) + \frac{1}{2} e^x \cos x.$$

(2) Solve $(D^2 - 5D + 6)y = xe^{4x}$.
Solution. The auxiliary equation of this differential equation is

$$\begin{aligned} m^2 - 5m + 6 &= 0 \\ (m-2)(m-3) &= 0 \\ m = 2, \quad m &= 3. \end{aligned}$$

Thus complementary function is

$$C.F = c_1 e^{2x} + c_2 e^{3x}.$$

Next, particular integral is given by

$$\begin{aligned} P.I &= \frac{1}{D^2 - 5D + 6} xe^{4x} \\ &= e^{4x} \frac{1}{(D+4)^2 - 5(D+4) + 6)} x \\ &= e^{4x} \frac{1}{D^2 + 3D + 2} x \end{aligned}$$

$$\begin{aligned}
&= e^{4x}\frac{1}{2\left(1+\frac{3D}{2}+\frac{D^2}{2}\right)}x\\
&= \frac{e^{4x}}{2}\left(1+\frac{3D}{2}+\frac{D^2}{2}\right)^{-1}x\\
&= \frac{e^{4x}}{2}\left(1-\frac{3D}{2}-\frac{D^2}{2}-\dots\right)x\\
&= \frac{e^{4x}}{2}\left(x-\frac{3}{2}\right).
\end{aligned}$$

Thus complete solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{e^{4x}}{2}\left(x-\frac{3}{2}\right) \quad \blacksquare$$

Show that **Rule VI**

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left(\frac{d}{dD}\frac{1}{f(D)}\right)V$$

where V is a function of x.
Solution. Let X be a function of x. Then by Leibnitz's theorem, we have

$$\begin{aligned}
D^n(xX) &= (D^nX)x + \binom{n}{1}(D^{n-1}X)1\\
&= xD^nX + nD^{n-1}X\\
&= xD^nX + \left(\frac{d}{dD}D^n\right)X\\
\Rightarrow f(D)(xX) &= xf(D)X + \left(\frac{d}{dD}f(D)\right)X. \qquad (1.13)
\end{aligned}$$

By putting $f(D)X = V$, we have

$$X = \frac{1}{f(D)}V.$$

Since X is a function of x, so is V. From (1.13)

$$f(D)\left(x\frac{1}{f(D)}V\right) = xV + \left(\frac{d}{dD}f(D)\right)\frac{1}{f(D)}V,$$

operating both sides by $f(D)$, we get

$$\begin{aligned}
\left(x\frac{1}{f(D)}V\right) &= \frac{1}{f(D)}(xV) + \frac{f'(D)}{(f(D))^2}V\\
\frac{1}{f(D)}(xV) &= x\frac{1}{f(D)}V - \frac{f'(D)}{(f(D))^2}V\\
\Rightarrow \frac{1}{f(D)}(xV) &= x\frac{1}{f(D)}V + \frac{d}{dD}\left(\frac{1}{f(D)}\right)V \quad \blacksquare
\end{aligned}$$

Example

Solve $(D^2+4)y = x\sin x$.
Solution. The auxiliary equation is

$$m^2+4=0 \quad \Rightarrow \quad m=\pm 2i.$$

Therefore, complementary function is

$$\begin{aligned} C.F &= c_1e^{2ix}+c_2e^{-2ix} \\ &= A\cos 2x + B\sin 2x. \end{aligned}$$

Now, particular integral is given as

$$\begin{aligned} P.I &= \frac{1}{D^2+4}x\sin x \\ &= x\frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}\sin x \\ &= x\frac{1}{-(1)^2+4}\sin x - \frac{2D}{(-(1)^2+4)^2}\sin x \\ &= \frac{1}{3}x\sin x - \frac{2}{9}D(\sin x) \\ &= \frac{1}{3}x\sin x - \frac{2}{9}\cos x. \end{aligned}$$

Therefore, complete solution is

$$y = A\cos 2x + B\sin 2x + \frac{1}{3}x\sin x - \frac{2}{9}\cos x \quad \blacksquare$$

1.2 Homogeneous linear differential equations

An equation of the form

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}x\frac{dy}{dx} + a_ny = X,$$

where $a_0, a_1, \ldots, a_n$ are all constants and X is a function of x, is called *Homogeneous linear differential equation* of nth order.

Solution of Homogeneous linear differential equation

Let $x=e^z$ then $z=\log x$, therefore

$$\frac{dz}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dy}{dx} = \frac{dy}{dz}\times\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$
$$\text{or} \quad x\frac{dy}{dx} = \frac{dy}{dz} \quad \Rightarrow \quad xDy = D'y \quad \text{where} \quad D=\frac{d}{dx} \quad \& \quad D'=\frac{d}{dz}.$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$
\begin{aligned}
&= \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} \\
&= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right) \\
\text{or} \quad x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz} \\
x^2D^2y = D'(D'-1)y.
\end{aligned}
$$

Similarly, one can prove that

$$
\begin{aligned}
x^3D^3y &= D'(D'-1)(D'-2)y \\
&\vdots
\end{aligned}
$$

Therefore by this suitable substitution the above homogeneous linear differential equation gets reduced to linear differential equation with constant coefficients ■

Example

Solve the following differential equation

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}.$$

Solution. In symbolic form the differential equation can be written as

$$x^2D^2y - 2y = x^2 + \frac{1}{x}.$$

Put $x = e^z$ then

$$xDy = D'y \quad \& \quad x^2D^2y = D'(D'-1)y.$$

This implies

$$
\begin{aligned}
D'(D'-1)y - 2y &= e^{2z} + \frac{1}{e^z} \\
(D'^2 - D' - 2)y &= e^{2z} + e^{-z}.
\end{aligned}
$$

This is linear differential equation with constant coefficients. Hence its auxiliary equation is

$$m^2 - m - 2 = 0 \quad \Rightarrow \quad m = 2, -1.$$

Therefore, complementary function is

$$
\begin{aligned}
C.F &= c_1e^{-z} + c_2e^{2z} \\
&= c_1(e^z)^{-1} + c_2(e^z)^2 \\
&= c_1(x)^{-1} + c_2x^2.
\end{aligned}
$$

Now particular integral is

$$P.I = \frac{1}{D'^2 - D' - 2}\left(e^{2z} - e^{-z}\right)$$

$$
\begin{aligned}
&= \frac{1}{(D'-2)(D'+1)}e^{2z} + \frac{1}{(D'-2)(D'+1)}e^{-z} \\
&= \frac{1}{3}\frac{1}{D'-2}e^{2z} + \frac{1}{-3}\frac{1}{D'+1}e^{-z} \\
&= \frac{1}{3}\left(z(e^z)^2 - z(e^z)^{-1}\right) \\
&= \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x.
\end{aligned}
$$

Thus complete solution is

$$y = c_1(x)^{-1} + c_2x^2 + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x \quad \blacksquare$$

1.3 Equation solvable for p

Suppose the equation of nth degree in p is $f(x, y, p) = 0$. As it is solvable for p. We can put it into the form

$$[p - F_1(x, y)]\,[p - F_2(x, y)] \ldots [p - F_n(x, y)] = 0.$$

Then, we solve each factors individually to get the solution of given differential equation ■

Example

Solve the differential equation $p^2 - p = 6$.
Solution. We have

$$
\begin{aligned}
p^2 + p - 6 &= 0 \\
(p-2)(p+3) &= 0 \\
\Rightarrow\ p = 2 \ \text{ or } \ p &= -3 \\
\text{i.e., } \frac{dy}{dx} = 2 \quad \text{or} \quad \frac{dy}{dx} &= -3.
\end{aligned}
$$

Integrating both sides of these two equations, we get

$$y = 2x + c_1 \qquad \text{and} \qquad y = -3x + c_2.$$

These two equations together gives the complete solution of given differential equation ■

Equation solvable for y

If the equation $f(x, y, p) = 0$ is solvable for y, we can express y explicitly in terms of x and p. Then, an equation solvable for y can be expressed as

$$y = \varphi(x, p). \tag{1.14}$$

Differentiating both sides of this equation with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\varphi(x, p)\right) = F\left(x, p, \frac{dp}{dx}\right)$$

$$\text{i.e.,}\quad p = F\left(x, p, \frac{dp}{dx}\right)$$

which is a differential equation involving two variables x and p. Let its solution be

$$g(x, p, c) = 0. \tag{1.15}$$

Therefore on eliminating p between (1.14) and (1.15), we get the required solution of (1.14). Similar is the case for the differential equations solvable for x.

1.4 Clairaut's equation

A differential equation of the form

$$y = px + f(p) \tag{1.16}$$

is known as Clairaut's equation.
To solve this differential equation we differentiate both sides of it with respect to x and we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \tag{1.17}$$

$$p = p + (x + f'(p))\frac{dp}{dx} \tag{1.18}$$

$$\Rightarrow \quad \frac{dp}{dx} = 0. \tag{1.19}$$

Integrating, we get

$$p = c, \quad \text{where } c \text{ is a constant.} \tag{1.20}$$

Eliminating p between (1.16) and (1.20), we get

$$y = cx + f(c)$$

that is the solution of given differential equation ■

1.5 Legendre Polynomials

Legendre Equation

The differential equation of the form

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1.21}$$

is called *Legendre's equation*, where n is a positive integer. We solve this differential equation by the method of *series solution* (also called *Frobenius method*). In this method we find a series representation for the solution instead of the solution itself. You first saw something like this when you looked at Taylor series in your Calculus class. The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y = \sum_{m=0}^{\infty} a_m x^{k-m}. \tag{1.22}$$

Let us assume (1.22) be the series solution of (1.21).Differentiating (1.22) and then putting the values of y, y' and y'' into (1.21), we have

$$(1-x^2)\sum_{m=0}^{\infty} a_m(k-m)(k-m-1)x^{k-m-2}$$
$$-2x\sum_{m=0}^{\infty} a_m(k-m)x^{k-m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^{k-m} = 0$$

or

$$\sum_{m=0}^{\infty} a_m\big[(k-m)(k-m-1)x^{k-m-2}$$
$$+\{n(n+1)-(k-m)(k-m-1)-2(k-m)\}\,x^{k-m}\big] = 0$$

or

$$\sum_{m=0}^{\infty} a_m\big[(k-m)(k-m-1)x^{k-m-2}$$
$$+\{n(n+1)-(k-m)(k-m+1)\}\,x^{k-m}\big] = 0$$

or, equivalently

$$\sum_{m=0}^{\infty} a_m\big[(k-m)(k-m-1)x^{k-m-2}$$
$$+(n-k+m)(n+k-m+1)x^{k-m}\big] = 0. \tag{1.23}$$

Since (1.23) is an identity, we can equate to zero the coefficients of various powers of x.

Therefore, equating to zero the coefficient of highest power of x, namely x^k in (1.23) and obtain

$$a_0(k-n)(k+n+1)=0 \quad \text{or} \quad (k-n)(k+n+1)=0.$$

Now $a_0 \neq 0$ as it is the coefficient of the first term with which we start to write series. Therefore

$$\text{either} \quad k=n \quad \text{or} \quad k=-(n+1). \tag{1.24}$$

The next lower power of x is $k-1$. So we equate to zero the coefficient of the highest power of x, namely x^k in (1.23) and obtain

$$a_1(k-1-n)(k+n)=0. \tag{1.25}$$

For $k=n$ or $k=-(n+1)$ neither $(k-1-n)$ nor $(k+n)$ is zero. So from (1.25), $a_1=0$. Finally, equating to zero the coefficient of the general term i.e., x^{k-m}, in (1.23), we have

$$a_{m-2}(k-m+2)(k-m+1)+(n-k+m)(n+k-m+1)a_m=0.$$

This gives

$$a_m = -\frac{(k-m+2)(k-m+1)}{(n-k+m)(n+k-m+1)}a_{m-2}. \tag{1.26}$$

Putting $m=3,5,7,\ldots$ in (1.26) and noting that $a_1=0$, we have

$$a_1=a_3=a_5=\ldots=0. \tag{1.27}$$

To obtain $a_2, a_4, \ldots$ etc., we consider two cases.
CASE I. When $k = n$, then (1.26) becomes

$$a_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} a_{m-2}. \tag{1.28}$$

Putting $m = 2, 4, 6, \ldots$ in (1.28), we have

$$\begin{aligned} a_2 &= -\frac{n(n-1)}{2(2n-1)} a_0 \\ a_4 &= -\frac{(n-2)(n-3)}{4(2n-3)} a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} a_0 \end{aligned}$$

and so on. Re-writing (1.22), for $k = n$ and replacing a_0 by a, we have

$$\begin{aligned} y &= a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \ldots \\ &= a \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - \ldots \right] \end{aligned} \tag{1.29}$$

which is one solution of Legendre's equation.
CASE II. When $k = -(n+1)$, then (1.26) becomes

$$a_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} a_{m-2}. \tag{1.30}$$

Putting $m = 2, 4, 6, \ldots$ in (1.30), we have

$$\begin{aligned} a_2 &= \frac{(n+1)(n+2)}{2(2n+3)} a_0 \\ a_4 &= -\frac{(n+3)(n+4)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} a_0 \end{aligned}$$

and so on. For $k = -(n+1)$, (1.22) gives with replacing a_0 by b

$$\begin{aligned} y &= a_0 x^{-(n+1)} + a_2 x^{-(n+3)} + a_4 x^{-(n+5)} + \ldots \\ &= b \Big[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} \\ &\quad + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \ldots \Big] \end{aligned} \tag{1.31}$$

which is another solution of Legendre's equation. Thus, two independent solutions of (1.21) are given by (1.29) & (1.31).

1.6 Legendre's function

If we take $a = [1.3.5 \ldots (2n-1)]/n!$, the solution (1.29) is denoted by $P_n(x)$ and is called Legendre's function of the first kind. Notice that (1.29) is a terminating series and so it gives rise to a polynomial of degree n called as Legendre's polynomial of degree n. Thus $P_n(x)$ is a particular solution of (1.21). We can write

$$P_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(2n-2m)!}{2^n.m!(n-2m)!(n-m)!} x^{n-2m}$$

where

$$[n/2] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}.$$

Again, if we take $b = n!/[1.3.5\dots(2n+1)]$, the solution (1.31) is denoted by $Q_n(x)$ and is called Legendre's function of second kind. Since n is a positive integer, (1.31) is an infinite or non-terminating series and hence $Q_n(x)$ is not a polynomial. We can write

$$Q_n(x) = \frac{n!}{[1.3.5\dots(2n+1)]}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5} + \dots\right].$$

The most general solution of the Legendre's equation is

$$y = AP_n(x) + BQ_n(x)$$

where A and B are arbitrary constants.

1.7 Exercise

Show that

(i) $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

(ii) $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

1.8 Generating function for Legendre polynomials

The Legendre polynomial $P_n(x)$ is the coefficient of z^n in the expansion in ascending powers of z of $(1 - 2xz + z^2)^{-1/2}$, i.e.,

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(z), \qquad |x| \leq 1, |z| < 1.$$

Proof. By Binomial theorem,

$$\begin{aligned}(1 - 2xz + z^2)^{-1/2} &= (1 - z(2x - z))^{-1/2} \\ &= 1 + \frac{1}{2}z(2x - z) + \frac{1.3}{2.4}z^2(2x - z)^2 \\ &\quad + \dots + \frac{1.3\dots(2n-3)}{2.4\dots(2n-2)}z^{n-1}(2x - z)^{n-1} \\ &\quad + \frac{1.3\dots(2n-1)}{2.4\dots(2n)}z^n(2x - z)^n + \dots\end{aligned}$$

Now coefficient of z^n in $\frac{1.3\dots(2n-1)}{2.4\dots(2n)}z^n(2x - z)^n$

$$= \frac{1.3\dots(2n-1)}{2.4.6\dots(2n)}(2x)^n$$

$$= \frac{1.3\ldots(2n-1)2^n x^n}{(2.1)(2.2)\ldots(2.n)}$$
$$= \frac{1.3\ldots(2n-1)2^n}{2^n n!}x^n$$
$$= \frac{1.3\ldots(2n-1)}{n!}x^n. \tag{1.32}$$

Again the coefficient of z^n in $\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}z^{n-1}(2x-z)^{n-1}$

$$= \frac{1.3\ldots(2n-3)}{(2.1)(2.2)\ldots 2(n-1)}\left\{-(n-1)(2x)^{n-2}\right\}$$
$$= -\frac{1.3\ldots(2n-3)}{2^{n-1}1.2\ldots(n-1)}.\frac{2n-1}{n}\frac{n}{2n-1}[(n-1)2^{n-2}\times x^{n-2}]$$
$$= -\frac{1.3\ldots(2n-1)}{n!}\frac{n(n-1)}{2(2n-1)}x^{n-2} \tag{1.33}$$

and so on. Using (1.32), (1.33),... we see that the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$, is given by

$$\frac{1.3\ldots(2n-1)}{n!}\Big[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\Big]$$

i.e., $P_n(x)$, by definition Legendre polynomial. Thus we find that $P_1(x), P_2(x), \ldots$ will be the coefficients of $z, z^2, \ldots$ in the expansion of $(1-2xz+z^2)^{-1/2}$. Thus we may write

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + \ldots \quad \blacksquare$$

Exercise

Show that

(1) $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$.

(2) $nP_n = xP'_n - P'_{n-1}$

(3) $(2n+1)P_n = P'_{n+1} - P'_{n-1}$.

Bessel's differential equation

The differential equation

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \tag{1.34}$$

is called Bessel's differential equation. We can solve this differential equation by the method of *series solution* and can prove that its general solution is given by

$$AJ_n(x) + BJ_{-n}(x)$$

where

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\Gamma(n+r+1)}$$

and $J_{-n}(x)$ is obtained by replacing n by $-n$ in $J_n(x)$.

Exercise

Show that

(1) $nJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$.

(2) $xJ'_n(x) = -nJ'_n + xJ_{n+1}(x)$.

(3) $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$.

Note: The exercises in next two sections are taken from "*Integral Calculus*" by Kochar, Chopra and Auzeem.

1.9 Integration of irrational functions

Exercise XX

Q. no 1

$$\text{Let} \quad I = \int \frac{\sqrt{x}}{x+1} dx.$$

Put $x = z^2$ then $dx = 2zdz$, we have

$$\begin{aligned} I &= \int \frac{z(2z)dz}{z^2+1} = 2\int \frac{z^2}{z^+1} dz \\ &= 2\int \frac{z^2+1-1}{z^2+1} dz = 2\int \left(2 + \frac{1}{z^2+1}\right) dz \\ &= 2(z - \tan^{-1} z) = 2\sqrt{x} - 2\tan^{-1}(\sqrt{x}) + c \quad \blacksquare \end{aligned}$$

Q. no. 5

$$\text{Let} \quad I = \int \frac{dx}{(1+x)^{1/2} - (1+x)^{1/3}}.$$

As $L.C.M(2,3) = 6$, put $1 + x = z^6$ then $dx = 6z^5 dz$

$$\begin{aligned} I &= \int \frac{6z^5}{z^3 - z^2} dz = 6\int \frac{z^3}{z-1} dz \\ &= 6\int \left\{(z^2 + z + 1) + \frac{1}{z-1}\right\} dz = 2z^3 + 3z^2 + 6z + \log(z-1) + c \end{aligned}$$

$$= 2(1+x)^{1/2} + 3(1+x)^{1/3} + 6(1+x)^{1/6} + 6\log\left[(1+x)^{1/6} - 1\right] + c \quad \blacksquare$$

Q. no. 8 Let

$$I = \int \frac{dx}{(1-x)\sqrt{1+x}}.$$

Put $1+x = z^2$ then we have $dx = 2zdz$.

$$\begin{aligned} I &= \int \frac{2zdz}{(1-z^2+1)z} = 2\int \frac{dz}{2-z^2} \\ &= 2\left(\frac{-1}{2\sqrt{2}}\right)\log\left(\frac{\sqrt{2}-z}{\sqrt{2}+z}\right) + c = -\frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}-\sqrt{1+x}}{\sqrt{2}+\sqrt{1+x}}\right) + c \quad \blacksquare \end{aligned}$$

Exercise XXI

Q. no. 4 Let

$$I = \int \frac{dx}{(x^2+1)\sqrt{x}}.$$

Put $x = z^2 \quad \Rightarrow \quad dx = 2zdz$

$$\begin{aligned} I &= \int \frac{2zdz}{(z^4+1)z} = \int \frac{2dz}{z^4+1} \\ &= \int \frac{(z^2+1)-(z^2-1)}{z^4+1}dz. \end{aligned}$$

Dividing numerator and denominator by z^2, we get

$$\begin{aligned} I &= \int \frac{1+\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz - \int \frac{1-\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz \\ &= \int \frac{1+\frac{1}{z^2}}{\left(z-\frac{1}{z}\right)^2+2}dz - \int \frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2-2}dz = I_1 + I_2. \end{aligned}$$

where

$$I_1 = \int \frac{1+\frac{1}{z^2}}{\left(z-\frac{1}{z}\right)^2+2}dz \quad \& \quad I_2 = \int \frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2-2}dz.$$

Put $z - 1/z = t$ in T_1 and we have $(1+1/z^2)dz = dt$

$$\begin{aligned} I_1 &= \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right). \end{aligned}$$

Next, put $z + 1/z = u$ in I_2 and $(1-\frac{1}{z^2})dz = du$, we get

$$I_2 = \int \frac{du}{u^2-2} = \int \frac{du}{u^2-(\sqrt{2})^2}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{2}} \log\left(\frac{u-\sqrt{2}}{u+\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} \log\left(\frac{z^2-\sqrt{2}z+1}{z^2+\sqrt{2}z+1}\right) \\
&= \frac{1}{2\sqrt{2}} \log\left(\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right).
\end{aligned}
$$

Thus,

$$
I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right) + \frac{1}{2\sqrt{2}} \log\left(\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right) + c \quad \blacksquare
$$

Q. no. 8 Let

$$
I = \int \frac{dx}{(x-1)\sqrt{x^2+1}}.
$$

Put $(x-1) = 1/z \quad \Rightarrow \quad dx = -1/x^2 dz$.

$$
\begin{aligned}
I &= \int \frac{1}{\frac{1}{z}\sqrt{\left(\frac{1}{z}+1\right)^2+1}}.\left(-\frac{1}{z^2}\right) dz \\
&= -\int \frac{dz}{z\sqrt{\frac{1}{z^2}+\frac{2}{z}+2}} = -\int \frac{dz}{\sqrt{2z^2+2z+1}} \\
&= -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{z^2+z+\frac{1}{2}}} = -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)^2+\frac{1}{4}}} \\
&= -\frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{z+\frac{1}{2}}{\frac{1}{2}}\right) + c = -\frac{1}{\sqrt{2}} \sinh^{-1}(2z+1) + c \\
&= -\frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{2}{x-1}+1\right) + c = -\frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{x+1}{x-1}\right) + c \quad \blacksquare
\end{aligned}
$$

Q. no. 13 Let

$$
I = \int \frac{dx}{(3x-4x^2)\sqrt{4-3x^2}}.
$$

Put $x = 1/y \quad \Rightarrow \quad dx = -1/y^2 dy$

$$
\begin{aligned}
I &= \int \frac{1}{\left(3+\frac{4}{y^2}\right)\sqrt{4-\frac{3}{y^2}}} \left(-\frac{1}{y^2} dy\right) \\
&= -\int \frac{1}{\left(\frac{3y^2+4}{y^2}\right)\sqrt{\frac{4y^2-3}{y^2}}} \frac{dy}{y^2} \\
&= -\int \frac{y dy}{(3y^2+4)\sqrt{4y^2-3}}.
\end{aligned}
$$

Put $\sqrt{4y^2+4} = z$, this implies

$$
4y^2 - 3 = z^2 \qquad \text{or} \quad y^2 \frac{z^2+3}{4}
$$

$$8ydy = 2zdz, \quad ydy = \frac{1}{4}zdz.$$

Now,

$$\begin{aligned}
I &= -\frac{1}{4}\int \frac{zdz}{\left\{3\left(\frac{z^2+3}{4}\right)+4\right\}z} \\
&= -\int \frac{dz}{(\sqrt{3}z)^2+(5)^2} \\
&= -\frac{1}{5}\frac{\tan^{-1}\left(\frac{\sqrt{3}z}{5}\right)}{\sqrt{3}}+c \\
&= -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{4y^2-3}\right)+c \\
&= -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{12y^2-9}}{5}\right)+c \\
&= -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{\frac{12}{x^2}-9}}{5}\right)+c \\
&= -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{12-9x}}{5x}\right)+c \quad \blacksquare
\end{aligned}$$

Exercise XXII

Q. no. 1 Let

$$I = \int \frac{dx}{x\sqrt{1+x^2}}.$$

Put $x = \tan\theta$, $dx = \sec^2\theta d\theta$.

$$\begin{aligned}
I &= \int \frac{\sec^2\theta d\theta}{\tan\theta\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2 d\theta}{\tan\theta.\sec\theta} \\
&= \int \frac{\sec\theta d\theta}{\tan\theta} = \int \text{cosec}\theta d\theta \\
&= \log\left(\text{cosec}\theta - \cot\theta\right)+c \\
&= \log\left(\sqrt{1+\cot^2\theta}-\cot\theta\right)+ \\
&= \log\left(\sqrt{1+\frac{1}{x^2}}-\frac{1}{x}\right)+c \\
&= \log\left(\frac{\sqrt{x^2+1}-1}{x}\right)+c \quad \blacksquare
\end{aligned}$$

Q. no. 7 Let

$$I = \int_a^{\sqrt{2}a} \frac{\sqrt{x^2-a^2}}{a}dx.$$

Put $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta$
when $x = a$, we have $a\sec\theta = a$ or $\theta = 0$
when $x = \sqrt{2}a$, we have $a\sec\theta = a\sqrt{2}$ or $\theta = \frac{\pi}{4}$.

$$\begin{aligned} I &= \int_0^{\pi/4} \frac{\sqrt{a^2\sec^2\theta - a^2}}{a\sec\theta}.a\sec\theta\tan\theta d\theta \\ &= \int_0^{\pi/4} a\tan\theta\tan\theta d\theta = a\int_0^{\pi/4}\tan^2\theta d\theta \\ &= a\int_0^{\pi/4}\left(sec^2\theta - 1\right)d\theta = a\left[\tan\theta - \theta\right]_0^{\pi/4} \\ &= a\left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)\right] \\ &= \left[1 - \frac{\pi}{4}\right]. \end{aligned}$$

Q. no. 10 Let

$$I = \int_0^{\pi/2} \frac{adx}{\left\{x + \sqrt{a^2 - x^2}\right\}^2}.$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$. Limits change from 0 to $\pi/2$.

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{a^2\cos\theta d\theta}{\left\{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}\right\}^2} \\ &= \int_0^{\pi/2} \frac{a^2\cos\theta d\theta}{a^2\left\{\sin\theta + \cos\theta\right\}^2} \\ &= \int_0^{\pi/2} \frac{\cos d\theta}{(\sin\theta + \cos\theta)^2}. \end{aligned} \tag{1.35}$$

Using properties of definite integrals, we get

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\cos(\pi/2 - \theta)\,d\theta}{\left[\sin(\pi/2 - \theta) + \cos(\pi/2 - \theta)\right]^2} \\ &= \int_0^{\pi/2} \frac{\sin\theta}{(\sin\theta + \cos\theta)^2}. \end{aligned} \tag{1.36}$$

Adding (1.35) and (1.36), we get

$$2I = \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{(\sin\theta + \cos\theta)^2}d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sin\theta + \cos\theta} d\theta$$
$$= \int_0^{\pi/2} \frac{d\theta}{\frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} + \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}}$$
$$= \int_0^{\pi/2} \frac{\sec^2(\theta/2)\, d\theta}{2\tan(\theta/2) + 1 - \tan^2(\theta/2)}.$$

Put $\tan(\theta/2) = t$, $\sec^2(\theta/2)d\theta/2 = dt$ or $\sec^2(\theta/2)d\theta = 2dt$. Limits of integration change from 0 to 2π,

$$2I = \int_0^1 \frac{2dt}{2t + 1 - t^2}$$
$$\text{or} \quad I = \int_0^1 \frac{dt}{2t+1-t^2} = \int_0^1 \frac{dt}{2-(t-1)^2}$$
$$= \int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2} = \frac{1}{2\sqrt{2}} \left[\log \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right]_0^1$$
$$= \frac{1}{2\sqrt{2}} \left[\log \frac{\sqrt{2}}{\sqrt{2}} - \log \frac{\sqrt{2}-1}{\sqrt{2}+1}\right] = \frac{1}{2\sqrt{2}} \log\left(1+\sqrt{2}\right).$$

1.10 Reduction Formulae

Exercise XXIII

Q. no. 1 Let $I = \int \sin^7 dx$.
Solution. We have,

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx \quad \text{(Prove it).} \tag{1.37}$$

Put $n = 1$, we get

$$I = \int \sin^7 x dx = -\frac{1}{7}\sin^6 \cos x + \frac{6}{7}\int \sin^5 x dx$$
$$= -\frac{1}{7}\sin^6 x \cos x + \frac{6}{7}\left[-\frac{1}{5}\sin^4 \cos x + \frac{4}{5}\int \sin^3 x dx\right]$$
$$\text{(By use of of (1.37))}$$

$$I = -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x + \frac{24}{35}\int \cos^3 dx.$$

Put $n = 3$ in (1.37), we get

$$I = -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x + \frac{24}{35}\left[-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x dx\right]$$
$$= -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x + \frac{24}{35}\left[-\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x dx\right]$$

$$= -\frac{1}{7}\cos x\left[\sin^6 x + \frac{6}{5}\sin^4 x + \frac{8}{5}\sin^2 x + \frac{16}{5}\right] \quad \blacksquare$$

Q. no. 3 Let $C_n = \int_0^{\pi/2} \cos^n x dx$ then

$$C_n = \frac{n-1}{n} C_{n-2}. \tag{1.38}$$

Put $n = 6$ and using (1.38) repeatedly, we get

$$\begin{aligned} C_6 = \int_0^{\pi/2} \cos^6 x dx &= \frac{5}{6}\ C_4 = \frac{5}{6}.\frac{3}{4}\ C_2 = \frac{5}{6}.\frac{3}{4}.\frac{1}{2}\ C_0 \\ &= \frac{5}{6}.\frac{3}{4}.\frac{1}{2}\ \int_0^{\pi/2} 1 dx = \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{5\pi}{32} \quad \blacksquare \end{aligned}$$

Exercise XXIV

Q. no. 1 Let $I = \int \cos^4 x \sin^2 x dx$ or $I(2,4) = \int \sin^2 x \cos^4 x dx$.
Let us first of all find the connection between the integrals

$$\int \sin^m x \cos^n x dx \qquad \text{and} \qquad \int \sin^{m-2} x \cos^n x dx.$$

Put $P = \sin^{\lambda+1} x \cos^{\mu+1} x$ where $\lambda = \min(m, m-2) = m-2$ and $\mu = \min(n, n) = n$.
Therefore, $P = \sin^{m-1} x \cos^{n+1} x$, differentiating both sides with respect to x, we get

$$\begin{aligned} \frac{dP}{dx} &= (m-1)\sin^{m-2} x(\cos x)\cos^{n+1} x + (n+1)(\sin^{m-1} x)\cos^n x(-\sin x) \\ &= (m-1)\sin^{m-2} x \cos^{n+2} x - (n+1)\cos^n x \sin^m x \\ &= (m-1)\sin^{m-2} x \cos^n x(1-\sin^2 x) - (n+1)\sin^m x \cos^n x \\ &= (m-1)]\sin^{m-2}\cos^n x - (m-1)\sin^m \cos^n x - (n+1)\sin^m x \cos^n x \\ &= (m-1)\sin^{m-2} x \cos^n x - (m+n)\sin^m \cos^n x. \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} & P = (m-1)\int \sin^{m-2} x \cos^n x dx - (m+n)\int \sin^m x \cos^n x dx \\ \Rightarrow\ & \sin^{m-1} x \cos^{n+1} x = (m-1)\int \sin^{m-2} x \cos^n x dx - (m+n)\int \sin^m x \cos^n x dx \\ & (m+n)\int \sin^m x \cos^n x dx = -\sin^{m-1} x \cos^{n+1} x + (m-1)\int \sin^{m-2} x \cos^n x dx \\ \therefore\ & \int \sin^m x \cos^x dx = -\frac{1}{m+n}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x dx. \end{aligned}$$

Put $m = 2, n = 4$, we get

$$\begin{aligned} \int \sin^2 x \cos^4 x dx &= -\frac{\sin x \cos^5 x}{6} + \frac{1}{6}\int \cos^4 x dx \\ &= -\frac{\sin x \cos^4 x}{6} + \frac{1}{6}\left[\frac{1}{4}\cos^3 \sin x + \frac{3}{4}\int \cos^2 x dx\right] \end{aligned}$$

$$\left(\text{By using the reduction formula of} \int \cos^n x dx \text{ which is given as}\right.$$

$$\left.\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx\right)$$

$$= -\frac{\sin x \cos^5 x}{6} + \frac{1}{24}\cos^3 x \sin x + \frac{1}{8}\left[\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1.dx\right]$$

$$= -\frac{\sin x \cos^5 x}{6} + \frac{1}{24}\cos^3 x \sin x + \frac{1}{16}\cos x \sin x + \frac{1}{16}x$$

$$= \frac{1}{48}\sin x \cos x \left[-8\cos^4 x + 2\cos^2 x + 3x + 3\right] \quad \blacksquare$$

Q. no. 5 See article 7.31 we can show there

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)!!(n-1)!!}{(m+n)!!} \quad \text{where } m \text{ or } n \text{ or both are odd.}$$

Therefore,

$$\int_0^{\pi/2} \sin^5 x \cos^4 x dx = \frac{4!!\ 3!!}{(5+4)!!} = \frac{(4.2)(3.1)}{9.7.5.3.1} = \frac{8}{315} \quad \blacksquare$$

Q no. 9 Let

$$I = \int_0^{\infty} \frac{x^5 dx}{(1+x^2)^6}.$$

Put $x = \tan\theta$, $dx = \sec^2\theta d\theta$. Limits of integration changes from 0 to $\pi/2$,

$$I = \int_0^{\pi/2} \frac{\tan^5\theta \sec^2\theta}{(1+\tan^2\theta)^6} d\theta = \int_0^{\pi/2} \frac{\tan^5\theta \sec^2\theta}{(sec^2\theta)^6} d\theta = \int_0^{\pi/2} \frac{\tan^5\theta \sec^2\theta}{sec^{12}\theta} d\theta = \int_0^{\pi/2} \frac{\tan^5\theta}{\sec^{10}\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^5\theta}{\cos^5\theta}\cos^{10}\theta d\theta = \int_0^{\pi/2} \sin^5\theta \cos^5\theta d\theta = \frac{(4!!)(4!!)}{(10!!)} = \frac{(4.2)(4.2)}{(10.8.6.4.2)} = \frac{1}{60} \quad \blacksquare$$

Exercise XXV

Q n. 1 Let $I = \int \tan^3 dx$. By reduction formula, we have

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{(n-1)} - \int \tan^{n-2} x dx \quad \text{(Prove it).}$$

Therefore,

$$\int \tan^3 x dx = \frac{\tan^2 x}{2} - \int \tan x dx = \frac{\tan^2 x}{2} - \log(\sec x) + c \quad \blacksquare$$

Q. no. 9 Let

$$I_n = \int_0^{\pi/4} \tan^n x dx.$$

We have by reduction formula,

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{(n-1)} - \int \tan^{n-2} x dx.$$

Therefore,

$$\begin{aligned} I_n &= \left.\frac{\tan^{n-1} x}{(n-1)}\right|_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x dx = \frac{1}{n-1}\left[\tan^{n-1}(\pi/4) - \tan^{n-1}(0)\right] - I_{n-2} \\ &= \frac{1}{n-1}(1-0) - I_{n-2} = \frac{1}{n-1} - I_{n-2} \\ &\Rightarrow (n-1)\left[I_n + I_{n-2}\right] = 1. \end{aligned}$$

Replacing $n-1$ by n, we get

$$n(I_{n+1} + I_{n-1}) = 1 \qquad \blacksquare$$

Exercise XXVI

Q. no. 1 Let

$$I = \int \frac{dx}{(x^2+a^2)^3}.$$

First of all let us find a connection between the integrals

$$\int x^0(a^2+x^2)^{-n} dx \qquad \& \qquad \int x^0(a^2+x^2)^{-n+1} dx.$$

To do this, let us put $P = x^{\lambda+1}(a^2+x^2)^{\mu+1}$ where λ is smaller of the two indices of x, i.e., $\lambda = 0$ and μ is the smaller of the two indices of (a^2+x^2), i.e., $\mu = -n$.
Differentiating P with respect to x, we get

$$\begin{aligned} \frac{dP}{dx} &= (a^2+x^2)^{-n+1} + x(-n+1)(a^2+x^2)^{-n}(2x) = (a^2+x^2)^{-n+1} + 2(-n+1)x^2(a^2+x^2)^{-n} \\ &= (a^2+x^2)^{-n+1} + 2(-n+1)(a^2+x^2-a^2)(a^2+x^2)^{-n} \\ &= (a^2+x^2)^{-n+1} + 2(-n+1)(a^2+x^2)^{-n+1} - 2a^2(-n+1)(a^2+x^2)^{-n} \\ &= (1-2n+2)(a^2+x^2)^{-n+1} + 2a^2(n-1)(a^2+x^2)^{-n} \\ \frac{dP}{dx} &= (3-2n)(a^2+x^2)^{-n+1} + 2a^2(n-1)(a^2+x^2)^{-n}. \end{aligned}$$

Integrating both sides with respect to x,

$$\begin{aligned} P &= (3-2n)\int (a^2+x^2)^{-n+1} dx + 2a^2(n-1)\int (a^2+x^2)^{-n} dx \\ x(a^2+x^2)^{-n+1} &= (3-2n)\int (a^2+x^2)^{-n+1} dx + 2a^2(n-1)\int (a^2+x^2)^{-n} dx \\ \frac{x}{(a^2+x^2)^{n-1}} &= (3-2n)\int \frac{dx}{(a^2+x^2)^{n-1}} + 2a^2(n-1)\int \frac{dx}{(a^2+x^2)^{n-1}} \\ \therefore \quad 2a^2(n-1)\int \frac{dx}{(a^2+x^2)^n} &= \frac{x}{(a^2+x^2)^{n-1}} + (2n-3)\int \frac{dx}{(a^2+x^2)^{n-1}} \end{aligned}$$

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2a^2(n-1)(a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} \int \frac{dx}{(a^2+x^2)^{n-1}}.$$

Put $n = 3, 2$ in succession, we get

$$\begin{aligned}
\int \frac{dx}{(a^2+x^2)^3} &= \frac{x}{4a^2(a^2+x^2)^2} + \frac{3}{4a^2} \int \frac{dx}{(a^2+x^2)^2} \\
&= \frac{x}{4a^2(a^2+x^2)^2} + \frac{3}{4a^2} \left[\frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^2} \int \frac{dx}{a^2+x^2} \right] \\
&= \frac{x}{4a^2(a^2+x^2)^2} + \frac{3}{4a^2} \left[\frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3} \tan^{-1}\left(\frac{x}{a}\right) \right] + c \qquad \blacksquare
\end{aligned}$$

Lecture 2

Fundamental Theorem Of Algebra
Every equation $f(x) = 0$, where $f(x)$ is a polynomial of degree n has a root real or complex.
Theorem.
An n^{th} degree equation in one variable cannot have more than n roots.
Proof. Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

By Fundamental Theorem of Algebra, this equation must have a root real or complex. Let α_1 be the root, so that $x - \alpha_1$ is a factor of $f(x)$. Thus

$$f(x) = (x - \alpha_1)f_1(x), \tag{0.1}$$

where $f_1(x)$ is a polynomial of degree $n - 1$.
Again $f_1(x) = 0$ must have a root α_2, so that $x - \alpha_2$ is a factor of $f_1(x)$. Thus

$$f_1(x) = (x - \alpha_2)f_2(x),$$

where $f_2(x)$ is a polynomial of degree $n - 2$.
Substituting the value of $f_1(x)$ in 0.1, we get

$$f(x) = (x - \alpha_1)(x - \alpha_2)f_2(x),$$

where $f_2(x)$ is a polynomial of degree $n - 2$.
Proceeding similarly at n^{th} stage we get

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n)f_n(x),$$

where $f_n(x)$ is a polynomial of degree $(n - n)$ i;e of zero degree. This means that $f_n(x)$ is a constant.

$$\therefore \quad a_0x^n + a_1x^{n-1} + \cdots + a_n = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n)f_n(x).$$

Comparing the coefficients of x^n, we get

$$a_0 = f_n(x)$$

$$\therefore \qquad f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n) = 0.$$

Which is satisfied by n values of x, i;e by $\alpha_1, \alpha_2, \cdots, \alpha_n$.
Hence $f(x) = 0$ has n roots.
Assume to the contrary that $f(x) = 0$ has more than n roots. Let β be any other root distinct from $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then

$$f(\beta) = a_0(\beta - \alpha_1)(\beta - \alpha_2)(\beta - \alpha_3)\cdots(\beta - \alpha_n).$$

Since $a_0 \neq 0$ and β is distinct from $\alpha_1, \alpha_2, \cdots, \alpha_n$, so $(\beta - \alpha_i) \neq 0,\ 1 \leq i \leq n$. Which means that $f(\beta) \neq 0$. Showing β is not a root of $f(x) = 0$. Which is a contradiction to our assumption that $f(x) = 0$ has more than n roots. Hence the equation $f(x) = 0$ of degree n cannot have more than n roots.

Synthetic Division
Let us divide the polynomial

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

by $x - h$ and find its quotient and the remainder. The quotient will be of the form

$$b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \cdots + b_{n-2}x + b_{n-1}$$

and let R be the remainder. Hence we get

$$\begin{aligned}a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n =&(x-h)b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \cdots + b_{n-2}x + b_{n-1} + R\\ =&b_0x^n + (b_1 - b_0h)x^{n-1} + (b_2 - b_1h)x^{n-2}\\ &+ \cdots + (b_{n-1} - b_{n-2}h)x + (R - b_{n-1}h)\end{aligned}$$

Equating the coefficients of $x^n, x^{n-1}, x^{n-2}, \cdots$ on both sides of the above expressions, we have

$$\begin{aligned}b_o =&a_0\\ b_1 - b_0h =&a_1\\ b_2 - b_1h =&a_2\\ \cdots&\cdots\\ \cdots&\cdots\\ b_{n-1} - b_{n-2}h =&a_{n-1}\\ R - b_{n-1}h =&a_n.\end{aligned}$$

This can be practically exhibited as follows

$$\begin{array}{cccccccc} h\surd a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ & \underline{b_0h} & \underline{b_1h} & \underline{b_2h} & \underline{\cdots} & \underline{b_{n-2}h} & \underline{b_{n-1}h} \\ a_0 & b_1 & b_2 & b_3 & \cdots & b_{n-1} & |\underline{R}. \end{array}$$

Where

$$\begin{aligned}b_0 =&a_0\\ b_1 =&a_1 + a_0h = a_1 + b_0h \qquad \text{as} \quad a_0 = b_0\\ b_2 =&a_2 + b_1h\\ b_3 =&a_3 + b_2h\\ \cdots&\cdots\\ b_{n-1} =&a_{n-1} + b_{n-2}h\\ R =&a_n + b_{n-1}h.\end{aligned}$$

Remark 1.

(i). *Multiplier h is the value of x obtained on putting the divisor $x - h$ equal to zero.*

(ii). *If we denote the original polynomial by $f(x)$ and the quotient by Q, then*

$$f(x) = (x-h)Q + R.$$

Putting $x = h$, we get

$$f(h) = R.$$

$\therefore$ *Remainder is also the value of the polynomial for $x = h$.*

(iii). *If $R = 0$, then the equation is exactly divisible by $(x - h)$ or the equation has a root h.*

(iv). *The missing coefficient or coefficients of the original polynomial (if there is any) should be replaced by zero or zeroes.*

Synthetic Division When The Divisor Is Another Polynomial

Let us divide $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ by $b_0x^2 + b_1x + b_2$ which is a quadratic expression. Suppose that the quotient is $c_0x^{n-2} + c_1x^{n-3} + c_2x^{n-4} + \cdots + c_{n-3}x + c_{n-2}$ and the remainder R is $Ax + B$. We can find the unknown coefficients $c_0, c_1, c_2, \cdots, c_{n-2}$ and A, B.

Example 1. *Divide* $x^3 - 6x^2 + 3x - 6$ *by* $(x-2), (x-3), (x+3)$.

Sol. Here $h = 2$, using Synthetic Division, it follows

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 11 & -6 \\ & & 2 & -8 & 6 \\ \hline & 1 & -4 & 3 & \underline{0} \end{array}$$

$\therefore$ Quotient $= x^2 - 4x + 3$ and $R = 0$.
Again $h = 3$, using Synthetic Division, we have

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & 3 & -9 & 6 \\ \hline & 1 & -3 & 2 & \underline{0} \end{array}$$

$\therefore$ Quotient $= x^2 - 3x + 2$ and $R = 0$.
Finally for $h = -3$, using Synthetic Division, we get

$$\begin{array}{r|rrrr} -3 & 1 & -6 & 11 & -6 \\ & & -3 & 27 & -114 \\ \hline & 1 & -9 & 38 & \underline{-120} \end{array}$$

$\therefore$ Quotient $= x^2 - 9x + 38$ and $R = -120$.

Lecture 7
Formation of Equations

Formation Of Equations Whose Roots Are The Functions Of The Roots Of A Given Equation

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the given equation

$$f(x) = 0. \tag{0.1}$$

It is required to find an equation whose roots are

$$\phi(\alpha_1), \phi(\alpha_2), \cdots, \phi(\alpha_n),$$

where $\phi(x)$ is an algebraic function of x.
Here $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $f(x) = 0$.
$\therefore \quad x = \alpha_1, \alpha_2, \cdots, \alpha_n$
Let $\quad y = \phi(\alpha_1), \phi(\alpha_2), \cdots, \phi(\alpha_n)$

$$\therefore \quad y = \phi(x). \tag{0.2}$$

Eliminating x between (0.1) and (0.2), we get $F(y) = 0$.

Equation Of Squared Difference

If α, β, γ be the roots of the equation $y^3 + 3Hy + G = 0$, to find an equation whose roots are

$$(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2, \alpha - \beta)^2.$$

Let $z = (\beta - \gamma)^2, (\gamma - \alpha)^2$
or we can write $z = (\beta + \gamma)^2 - 4\beta\gamma$, etc.
or $\quad z = (\alpha + \beta + \gamma - \alpha)^2 - \dfrac{4\alpha\beta\gamma}{\alpha}$, etc.
or $\quad z = (0 - \alpha)^2 + \dfrac{4G}{\alpha}$, etc.
or $\quad z = y^2 + \dfrac{4G}{y}$
or $\quad yz = y^3 + 4G.$
From original equation

$$y^3 = -3Hy - G. \tag{0.3}$$

Substituting this in (0.3), we get

$$yz = -3Hy - G + 4G$$

$$\text{or} \quad y = \frac{3G}{z + 3G}$$

Substituting this value of y in the original equation, we get

$$\frac{27G^3}{(z + 3H)^3} + \frac{9GH}{(z + 3H)} + G = 0$$

$$\text{or} \quad (z + 3H)^3 + 9H(z + 3H)^2 + 27G^2 = 0$$

$$z^3 + 18Hz + 81H^2z + 27(G^2 + 4H^3) = 0,$$

is the required equation.

Example 1. *If α, β, γ are the roots of the equation*

$$x^3 + 3x^2 + 2x + 1 = 0,$$

find the equation whose roots are

$$\beta + \gamma, \gamma + \alpha, \alpha + \beta.$$

Sol. The given equation is

$$x^3 + 3x^2 + 2x + 1 = 0 \tag{0.4}$$

Let $y = (\beta + \gamma), (\gamma + \alpha), (\alpha + \beta)$.
we take

$$\begin{aligned} y &= \beta + \gamma \\ \implies \quad y &= \alpha + \beta + \gamma - \alpha \\ \implies \quad y &= -3 - \alpha \qquad \text{since sum of roots} = \alpha + \beta + \gamma = -3 \\ \implies \quad y &= -3 - x \end{aligned}$$

$$\implies \quad x = -3 - y \tag{0.5}$$

Substituting (0.5) in (0.4), we get

$$\begin{aligned} (-3 - y)^3 + 3(-3 - y)^2 + 2(-3 - y) + 1 = 0 \\ -y^3 - 6y^2 - 11y + 5 = 0 \end{aligned}$$

is the required equation whose roots are $\beta + \gamma, \gamma + \alpha, \alpha + \beta$.

Example 2. *If α, β, γ are the roots of the equation $2x^3 + 3x^2 + x + 3 = 0$, find the equation whose roots are*

$$\alpha + \frac{1}{\beta\gamma}, \beta + \frac{1}{\gamma\beta}, \gamma + \frac{1}{\alpha\beta}.$$

Sol. The given equation is

$$2x^3 + 3x^2 + x + 3 = 0 \tag{0.6}$$

Let $y = \alpha + \frac{1}{\beta\gamma}, \beta + \frac{1}{\gamma\beta}, \gamma + \frac{1}{\alpha\beta}$. We take

$$\begin{aligned} y &= \alpha + \frac{1}{\beta\gamma} \\ \implies \quad y &= \alpha + \frac{\alpha}{\alpha\beta\gamma} \\ \implies \quad y &= \alpha + \frac{\alpha}{\frac{-3}{2}} \\ \implies \quad y &= \alpha + \frac{2\alpha}{-3} \\ \implies \quad y &= \frac{\alpha}{3} \end{aligned}$$

Taking

$$\begin{aligned} y &= \frac{x}{3} \\ \implies \quad x &= 3y \end{aligned} \tag{0.7}$$

Put (0.7) in (0.6), we get

$$2(3y)^3 + 3(3y)^2 + 3y = 0$$
$$\implies \quad 3(18y^3 + 9y^2 + y + 1) = 0$$
$$\implies \quad 18y^3 + 9y^2 + y + 1 = 0,$$

is the required equation whose roots are $\alpha + \dfrac{1}{\beta\gamma}, \beta + \dfrac{1}{\gamma\beta}, \gamma + \dfrac{1}{\alpha\beta}$.

Example 3. *If α, β, γ are the roots of the equation $x^3 - 3x + 2 = 0$, find the equation whose roots are*

$$\beta^3 + \gamma^3 - \alpha^3, \ etc.$$

Sol. The given equation is

$$x^3 - 3x + 2 = 0. \tag{0.8}$$

Let $y = \beta^3 + \gamma^3 - \alpha^3, etc.$ We take

$$y = \beta^3 + \gamma^3 - \alpha^3$$
$$\implies \quad y = \alpha^3 + \beta^3 + \gamma^3 - 2\alpha^3$$
$$\implies \quad y = 3\alpha\beta\gamma + \left(\sum \alpha\right)\left[\left(\sum \alpha\right)^2 - 3\left(\sum \alpha\beta\right)\right] - 2\alpha^3 \qquad \text{Since } \sum \alpha = 0, \quad \sum \alpha\beta = 3, \alpha\beta\gamma = -2$$
$$\implies \quad y = -6 - 2\alpha^3$$

Taking

$$y = -6 - 2\alpha^3$$
$$\implies \quad x = \left(\frac{y+6}{-2}\right)^{\frac{1}{3}} \tag{0.9}$$

Put (0.9) in (0.8), we get

$$\left(\left(\frac{y+6}{-2}\right)^{\frac{1}{3}}\right)^3 - 3\left(\frac{y+6}{-2}\right)^{\frac{1}{3}} + 2 = 0$$
$$\implies \quad \frac{y+6}{-2} + 2 = 3\left(\left(\frac{y+6}{-2}\right)^{\frac{1}{3}}\right)^3$$

Cubing both sides and simplifying we get

$$y^3 + 6y^2 - 96y - 640 = 0$$

is the required equation whose roots are $\beta^3 + \gamma^3 - alpha^3, etc.$

Lecture 3

Relation Between The Roots And The Coefficients Of An Equation

Let $\alpha, \beta, \gamma, \cdots, \rho$ be the roots of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

We can write this equation as

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$$

$$\begin{aligned}\therefore \qquad x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} =& (x-\alpha)(x-\beta)(x-\gamma)\cdots(x-\rho) \\ =& x^n - (\alpha+\beta+\gamma+\cdots+\rho)x^{n-1} + (\alpha\beta+\beta\gamma+\gamma\delta+\cdots)x^{n-2} \\ &- (\alpha\beta\gamma+\beta\gamma\delta\cdots)x^{n-3} + (\alpha\beta\gamma\delta+\cdots)x^{n-4} \\ &+ \cdots + (-1)^n(\alpha\beta\gamma\delta\cdots\rho).\end{aligned}$$

Equating the coefficients of the corresponding terms of the two sides, we get

$$\begin{aligned}\alpha+\beta+\gamma+\delta+\cdots+\rho =& \sum \alpha = -\frac{a_1}{a_0} \\ \alpha\beta+\beta\gamma+\gamma\delta+\cdots =& \sum \alpha\beta = +\frac{a_2}{a_0} \\ \alpha\beta\gamma+\beta\gamma\delta+\cdots =& \sum \alpha\beta\gamma = +\frac{a_3}{a_0} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ (-1)^n\alpha\beta\gamma\delta\cdots\rho =& \frac{a_n}{a_0} \\ \text{or} \qquad \alpha\beta\gamma\delta\cdots\rho =& (-1)^n\frac{a_n}{a_0}\end{aligned}$$

Particular cases

(a). If α, β, γ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$, then

$$\begin{aligned}\alpha+\beta+\gamma =& -\frac{b}{a} \\ \alpha\beta+\beta\gamma+\gamma\alpha =& \frac{c}{a} \\ \alpha\beta\gamma =& -\frac{d}{a}.\end{aligned}$$

(b). If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\begin{aligned}\alpha+\beta+\gamma+\delta =& -\frac{b}{a} \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta =& \frac{c}{a} \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta =& -\frac{d}{a} \\ \alpha\beta\gamma\delta =& \frac{e}{a}\end{aligned}$$

Corollary 1. *If a root be repeated r times in $f(x) = 0$, then this root will be repeated $(r-1)$ times in $f'(x) = 0$, where $f'(x)$ is derivative of $f(x)$.*

Proof: Let $f(x) = (x-\alpha)^r \phi(x)$, where α occurs r times in $f(x) = 0$ and $\phi(\alpha) \neq 0$. Therefore

$$\begin{aligned} f'(x) =& (x-r)^r \phi'(x) + r(x-\alpha)^{r-1}\phi(x) \\ =& (x-r)^{r-1}[(x-\alpha)\phi'(x) + r\phi(x)]. \end{aligned}$$

Evidently $f'(x) = 0$ has the root α repeated $(r-1)$ times.

Definition:(Arithmetic Progression)
Numbers $a_1, a_2, a_3, \cdots, a_n$ are said to be in Arithmetic Progression(A.P.) if

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_n - a_{n-1}$$
$$i.e. \quad \text{if} \quad a_{k+1} - a_k = d \text{ (constant)} \qquad \forall\, k, 1 \leq k \leq n.$$

This constant d is called the common difference. A general A.P. is $a, a+d, a+2d, a+3d, \cdots$.
Definition:(Geometric Progression)
Non-zero numbers $a_1, a_2, a_3, \cdots, a_n$ are said to be in Geometric Progression(G.P.), if

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = \frac{a_n}{a_n - 1}$$
$$i.e. \quad \text{if} \quad \frac{a_{k+1}}{a_k} = a \text{ (constant)} \qquad \forall\, k, 1 \leq k \leq n.$$

Definition:(Hyper Geometric Progression)
Non-zero numbers $a_1, a_2, a_3, \cdots, a_n$ are said to be in Harmonic Progression(H.P.), if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots, \frac{1}{a_n}$ are in A.P.

Example 1. *Solve the following equations:*

(1). $x^3 - 6x^2 + 11x - 6 = 0$, *if the roots are in A.P.*

(2). $x^3 - 7x^2 + 14x - 8 = 0$, *if the roots are in G.P.*

(3). $6x^3 - 11x^2 + 6x - 1 = 0$, *if the roots are in H.P.*

Sol.1. Let $\alpha - d, \alpha, \alpha + d$ be the roots of given equation $x^3 - 6x^2 + 11x - 6 = 0$.
$\therefore$ Sum of roots$= -\frac{b}{a}$
Hence $\alpha - d + \alpha + \alpha + d = 6$
$\implies \alpha = 2.$
So $\alpha = 2$ is one of the three roots of the given equation and other two roots can be obtained by synthetic division as

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 11 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & -4 & 3 & |\underline{0} \end{array}$$

$\therefore \quad x^2 - 4x + 3 = 0$
$\implies \quad (x-1)(x-3) = 0$
$\implies x = 1, 3$
$\therefore \quad x = 1, 2, 3$ are the three roots of the given equation which are in A.P..
Sol.2. Let $\frac{\alpha}{\beta}, \alpha, \alpha\beta$ be the roots of given equation $x^3 - 7x^2 + 14x - 8 = 0$.

$\therefore$ Product of roots$=-\dfrac{d}{a}$
Hence $\dfrac{\alpha}{\beta}.\alpha.\alpha\beta = 8$
$\implies (\alpha)^3 = 8.$
So $\alpha = 2$ is one of the three roots of the given equation and other two roots can be obtained by synthetic division as

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 14 & -8 \\ & & 2 & -10 & 8 \\ \hline & 1 & -5 & 4 & |\underline{0} \end{array}$$

$\therefore \quad x^2 - 5x + 4 = 0$
$\implies \quad (x-1)(x-4) = 0$
$\implies x = 1, 4$
$\therefore \quad x = 1, 2, 4$ are the three roots of the given equation which are in G.P..

Sol.3. Put $x = \dfrac{1}{y}$, the given equation becomes

$$\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0$$

$$\implies \quad -y^3 + 6y^2 - 11y + 6 = 0$$

$$\implies \quad y^3 - 6y^2 + 11y - 6 = 0$$

So the roots of the new equation are in A.P. Let $\alpha - d, \alpha, \alpha + d$ be the roots of given equation $y^3 - 6y^2 + 11y - 6 = 0$.
$\therefore$ Sum of roots$=-\dfrac{b}{a}$.
Hence $\alpha - d + \alpha + \alpha + d = 6$
$\implies \alpha = 2.$
So $\alpha = 2$ is one of the three roots of the given equation and other two roots can be obtained by synthetic division as

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 11 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & -4 & 3 & |\underline{0} \end{array}$$

$\therefore \quad y^2 - 4y + 3 = 0$
$\implies \quad (y-1)(y-3) = 0$
$\implies y = 1, 3$
$\therefore \quad y = 1, 2, 3$ are the three roots of the equation $y^3 - 6y^2 + 11y - 6 = 0$. Therefore $1, \dfrac{1}{2}, \dfrac{1}{3}$ are the roots of the given equation which are in H.P.

Example 2. *Solve* $4x^3 - 8x^2 + 5x - 1 = 0$, *given that it has a pair of equal roots.*

Sol: Since the given equation has a pair of equal roots. Let α, α and β be the roots.
$\therefore$ Sum of roots$=-\dfrac{b}{a}$.
Hence $\alpha + \alpha + \beta = -\left(-\dfrac{8}{4}\right) = 2$

$\implies 2\alpha + \beta = 2.$
$\implies \beta = 2 - 2\alpha.$
Also, we have

$$\begin{aligned}
\alpha.\alpha + \alpha.\beta + \beta.\alpha &= \frac{5}{4} \\
\implies \quad \alpha^2 + 2\alpha.\beta &= \frac{5}{4} \\
\implies \quad \alpha^2 + 2\alpha.(2 - 2\alpha) &= \frac{5}{4} \qquad \text{using } \beta = 2 - 2\alpha \\
\implies \quad \alpha^2 + 4\alpha - 4\alpha^2 &= \frac{5}{4} \\
\implies \quad -3\alpha^2 + 4\alpha &= \frac{5}{4} \\
\implies \quad 3\alpha^2 - 4\alpha + \frac{5}{4} &= 0 \\
\implies \quad 12\alpha^2 - 16\alpha + 5 &= 0 \\
\implies \quad (2\alpha - 1)(6\alpha - 5) &= 0 \\
\implies \quad \alpha &= \frac{1}{2}, \frac{5}{6}.
\end{aligned}$$

Taking $\alpha = \frac{1}{2}$, we get $\beta = 2 - 2\left(\frac{1}{2}\right) = 2 - 1 = 1.$
Using Synthetic Division we get

$$\begin{array}{c|cccc}
1 & 4 & -8 & 5 & -1 \\
 & & 4 & -4 & 1 \\
\hline
 & 4 & -4 & 1 & |\underline{0}
\end{array}$$

$\therefore \quad 4x^2 - 4x + 1 = 0$
$\implies \quad (2x - 1)(2x - 1) = 0$
$\implies x = \frac{1}{2}, \frac{1}{2}$
$\therefore \quad x = 1, \frac{1}{2}, \frac{1}{2}$ are the three roots of the given equation.

Example 3. *Find the conditions so that the roots of the given equation*

$$x^4 + px^3 + qx^2 - rx + s = 0$$

may be (a). A.P. (b). G.P. (c). H.P.

Sol.(a). The given equation is

$$x^4 + px^3 + qx^2 - rx + s = 0 \tag{0.1}$$

Since the roots are in A.P. Therefore equation (0.1) can be put in the form

$$\left(x^2+\frac{a}{2}x+A\right)\left(x^2+\frac{b}{2}x+B\right)=0$$

$$\implies \left(x^2-\frac{p}{2}x+A\right)\left(x^2-\frac{p}{2}x+B\right)=0$$

$$\implies x^4-\frac{2p}{2}x^3+\left(A+B+\frac{p^2}{2}\right)x^2-\frac{p}{2}(A+B)x+AB=0 \tag{0.2}$$

Equating coefficients of (0.1) and (0.2), we get

$$A+B+\frac{p^2}{2}=q, \qquad -\frac{p}{2}(A+B)=-r \quad \text{and} \quad AB=s.$$

$$\implies A+B=q-\frac{p^2}{2} \quad \text{and} \quad A+B=\frac{2r}{p}.$$

Comparing above two equations, we get

$$\frac{2r}{p}=q-\frac{p^2}{2}$$

$$\implies \frac{8r}{p}=4q-p^2$$

$$\implies p^3+8r=4pq$$

is the required condition for roots to be in A.P.

Sol.(b). Since the roots are in G.P. Therefore equation (0.1) can be put in the form

$$\left(x^2+Ax+\sqrt{d}\right)\left(x^2+Bx+\sqrt{d}\right)=0$$

Let α,β,γ and δ be the roots of given equation, then $\alpha\beta=K$ and $\gamma\delta=K$.

$$\left(x^2+Ax+K\right)\left(x^2+Bx+K\right)=0$$

$$x^4+(A+B)x^3+(2K+AB)x^2+K(A+B)x+K^2=0 \tag{0.3}$$

Equating the coefficients of (0.1) and (0.3), we get

$A+B=-p,\quad AB+2K=q,\quad K(A+B)=-r \implies A+B=\dfrac{-r}{K}$ and $K^2=s$.

From above we get

$$-p=-\frac{r}{K}$$

$$\implies p^2=\frac{r^2}{K^2}$$

$$\implies p^2s=r^2,$$

is the required condition for the roots to in G.P.

Sol.(c). It is given that the roots are in G.P. Put $x = \frac{1}{y}$, the given equation becomes

$$\frac{1}{y^4} - p\frac{1}{y^3} + \frac{q}{y^2} - r\frac{1}{y} + s = 0$$
$$\implies 1 - py + qy^2 - ry^3 + sy^4 = 0$$
$$\implies \quad sy^4 - ry^3 + qy^2 - py + 1 = 0 \tag{0.4}$$

The roots of (0.4) are in A.P. Thus (0.4) can be put in the form

$$\left(y^2 + \frac{a}{2}y + A\right)\left(y^2 + \frac{b}{2}y + B\right) = 0$$
$$\implies \left(y^2 - \frac{r}{2s}y + A\right)\left(y^2 - \frac{r}{2s}y + B\right) = 0$$
$$\implies y^4 - \frac{2r}{2s}y^3 + \left(A + B + \frac{r^2}{4s^2}\right)y^2 - \frac{r}{2s}(A+B)y + AB = 0 \tag{0.5}$$

Equating the coefficients of (0.4) and (0.5), we get

$$A + B + \frac{r^2}{4s^2} = \frac{q}{s}, \qquad \frac{r}{2s}(A+B) = \frac{p}{s} \quad \text{and} \quad AB = \frac{1}{s}.$$

$$\implies A + B = \frac{q}{s^2} - \frac{r^2}{4s^2} \qquad \text{and} \quad A + B = \frac{2ps}{rs}$$

Comparing above two equations, we get

$$\frac{q}{s} - \frac{r^2}{4s^2} = \frac{2ps}{sr}$$
$$\implies \frac{4qs^2 - sr^2}{4s^2} = \frac{2ps}{r}$$
$$\implies 4qs^2 - sr^2 = \frac{8ps^3}{r}$$
$$\implies 4qs^2 = sr^2 + \frac{8ps^3}{r}$$
$$\implies 4qs = r^2 + \frac{8ps^2}{r}$$
$$\implies 4qrs = r^3 + 8ps^2$$

is the required condition for roots to be in H.P.

Example 4. *The cubic equation $x^3 - 9x^2 + 12x + \lambda = 0$ has two equal roots. Find λ and solve the equation completely.*

Sol. The given equation is

$$x^3 - 9x^2 + 12x + \lambda = 0 \tag{0.6}$$

Since two of its roots are equal. Let α, α and β be the roots.

$\therefore$ Sum of roots$=-\frac{b}{a}$.

Hence $\alpha + \alpha + \beta = -\left(-\frac{9}{2}\right) = 2$
$\implies 2\alpha + \beta = \frac{9}{2}$.
$\implies \beta = \frac{9}{2} - 2\alpha$.
Also, we have

$$\begin{aligned} \alpha.\alpha + \alpha.\beta + \beta.\alpha &= \frac{12}{2} \\ \implies \alpha^2 + 2\alpha.\beta &= 6 \\ \implies \alpha^2 + 2\alpha.\left(\frac{9}{2} - 2\alpha\right) &= 6 \qquad \text{using } \beta = \frac{9}{2} - 2\alpha \\ \implies -3\alpha^2 + 9\alpha - 6 &= 0 \\ \implies 3\alpha^2 - 9\alpha + 6 &= 0 \\ \implies (\alpha - 2)(3\alpha - 3) &= 0 \\ \implies \alpha &= 1, 2. \end{aligned}$$

Case I. For $\alpha = 1$, we get $\beta = \frac{9}{2} - 2 = \frac{5}{2}$.
$\therefore$ Sum of roots $= -\frac{d}{a}$.
$\implies \alpha.\alpha.\beta = -\frac{\lambda}{2}$
$\implies 1^2.\frac{5}{2} = -\frac{\lambda}{2}$ using $\alpha = 1, \beta = \frac{5}{2}$.
$\implies \lambda = -5$.
Substituting the value of $\lambda = -5$ in the given equation we get

$$x^3 - 9x^2 + 12x - 5 = 0. \tag{0.7}$$

since $x = 1$ get satisfied by the equation (0.7), so $x = 1$ is root of (0.7) and the other two roots are obtained by using Synthetic Division as

$$\begin{array}{c|cccc} 1 & 2 & -9 & 12 & -5 \\ & & 2 & -7 & 5 \\ \hline & 2 & -7 & 5 & |\underline{0} \end{array}$$

$\therefore \quad 2x^2 - 7x + 5 = 0$
$\implies (x - 1)(2x - 5) = 0$
$\implies x = 1, \frac{5}{2}$.
Hence the roots are $x = 1, 1, \frac{5}{2}$.

Case II. For $\alpha = 2$, we get $\beta = \frac{9}{2} - 2(2) = \frac{9}{2} - 4 = \frac{1}{2}$.
Now proceed as in case I.

SYMMETRIC FUNCTIONS

Definition:

A symmetric function of the roots of an equation is that function in which the roots occur in such a way that the expression containing these roots remains unaltered when any two of these roots are interchanged.

Consider $f(\alpha, \beta, \gamma) = \alpha^3 + \beta^3 + \gamma^3 + \alpha\beta + \beta\gamma + \gamma\alpha$, is the example of a symmetric function.

For the sake of convenience we denote

$$\begin{aligned} \alpha^3 + \beta^3 + \gamma^3 \;\; &\text{by} \;\; \sum \alpha^3, \\ \alpha\beta + \beta\gamma + \gamma\alpha \;\; &\text{by} \;\; \sum \alpha\beta, \\ \alpha + \beta + \gamma \;\; &\text{by} \;\; \sum \alpha. \end{aligned}$$

Therefore above symmetric function can be written as $f(\alpha, \beta, \gamma) = \sum \alpha^3 + \sum \alpha\beta$.

We already know the relation between roots and coefficients of an equation, i,e. if $\alpha, \beta, \gamma, \cdots, \rho$ be the roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

then

$$\begin{aligned} \alpha + \beta + \gamma + \delta + \cdots + \rho =& \sum \alpha = -\frac{a_1}{a_0} \\ \alpha\beta + \beta\gamma + \gamma\delta + \cdots =& \sum \alpha\beta = +\frac{a_2}{a_0} \\ \alpha\beta\gamma + \beta\gamma\delta + \cdots =& \sum \alpha\beta\gamma = +\frac{a_3}{a_0} \\ \cdots\cdots\cdots&\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots&\cdots\cdots\cdots\cdots \\ \alpha\beta\gamma\delta\cdots\rho =& (-1)^n \frac{a_n}{a_0}. \end{aligned}$$

These symmetric functions are called **Elementary Symmetric Functions**.

Example 1. *If α, β, γ are the roots of the equation $x^3 - ax^2 + bx - c = 0$, find the values of*

(i). $\sum \alpha^2$.
(ii). $\sum \alpha^3$.

Sol. Here

$$\begin{aligned} \sum \alpha &= \alpha + \beta + \gamma = a \\ \sum \alpha\beta &= \alpha\beta + \beta\gamma + \gamma\alpha = b \\ \alpha\beta\gamma &= c. \end{aligned}$$

(i). We have $\sum \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = (\sum \alpha)^2 - \sum \alpha\beta = a^2 - 2b$.

(ii). Also

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$$

or

$$\begin{aligned}\alpha^3+\beta^3+\gamma^3 &= 3\alpha\beta\gamma+(\alpha+\beta+\gamma)\left[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\right]\\ &= 3\alpha\beta\gamma+\left(\sum\alpha\right)\left[\left(\sum\alpha\right)^2-3\left(\sum\alpha\beta\right)\right]\\ &= 3c+a(a^2-3b)\\ &= a^3-3ab+3c.\end{aligned}$$

Example 2. *If α,β,γ are the roots of the equation $x^3+px^2+qx+r=0$, find the value of*

(i). $\sum\frac{1}{\alpha}$.
(ii). $\sum\alpha^2$.

Sol. Here

$$\begin{aligned}\sum\alpha &= \alpha+\beta+\gamma=-p\\ \sum\alpha\beta &= \alpha\beta+\beta\gamma+\gamma\alpha=q\\ \alpha\beta\gamma &= -r.\end{aligned}$$

(i). We have

$$\begin{aligned}\sum\frac{1}{\alpha} &= \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\\ &= \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\\ &= \frac{\sum\alpha\beta}{\alpha\beta\gamma}\\ &= \frac{q}{-r}=-\frac{q}{r}.\end{aligned}$$

(ii). We have

$$\begin{aligned}\sum\alpha^2 &= \alpha^2+\beta^2+\gamma^2\\ &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)\\ &= (\sum\alpha)^2-\sum\alpha\beta\\ &= (-p)^2-2q=p^2-2q.\end{aligned}$$

Example 3. *If α,β,γ are the roots of the equation $x^3-px^2+qx-r=0$, find the value of*

(i). $\sum\alpha^2\beta$.
(ii). $\sum\alpha^2\beta\gamma$.

Sol. Here

$$\begin{aligned}\sum\alpha &= \alpha+\beta+\gamma=p\\ \sum\alpha\beta &= \alpha\beta+\beta\gamma+\gamma\alpha=q\\ \alpha\beta\gamma &= r.\end{aligned}$$

(i). We have

$$\sum \alpha^2\beta = \sum \alpha \sum \alpha\beta - 3\alpha\beta\gamma$$
$$= pq - 3r.$$

(ii). We have

$$\sum \alpha^2\beta\gamma = \alpha\beta\gamma \sum \alpha$$
$$= rp$$
$$= pr.$$

THEORY OF EQUATIONS

Lecture 1

Motivation:

It is folklore to find the roots of a $2nd$ degree polynomial $ax^2 + bx + c = 0$, the formula for which is given to be $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

We know that there is no such fundamental formula to find the roots of a polynomial whose degree is greater than 2. In these notes we will discuss some techniques to find roots of a Cubic and Bi-quadratic equations.

Definition 1. A polynomial $p(x)$ is defined as a mathematical expression of the form

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

where $a_0, a_1, \cdots, a_n$ are constants, $(n-k) \geq 0$ and n an integer greater than 1. If $a_0 \neq 0$, then the polynomial $p(x)$ is of degree n. Further, if we equate $p(x)$ to zero

$$\text{i;e } p(x) = 0,$$

this is called n^{th} degree equation in variable x.

Theorem. (**Remainder**)
If $f(x)$ is a polynomial, f(h) is the remainder when $f(x)$ is divided by $x - h$.
Proof: Let Q be the quotient and R be the remainder when $f(x)$ is divided by $x - h$.
We have

$$f(x) = (x-h)Q + R, \quad \text{where } R \text{ is a constant.}$$

Putting $x = h$, we have $f(h) = R$, which proves the theorem.

Definition 2. A number α(real or complex) is said to be the root of $p(x) = 0$, if $p(\alpha) = 0$.

Theorem. (**Factor**)
If h is a root of the equation $f(x) = 0$, $f(x)$ being a polynomial of n^{th} degree, then $(x-h)$ is a factor of $f(x)$ and conversely.
Proof: Let us divide $f(x)$ by $x - h$, so as to give Q as the quotient and R as the remainder

$$\therefore \qquad f(x) = (x-h)Q + R.$$

Putting $x = h$, we get

$$f(h) = R.$$

But $\qquad f(h) = 0$, as h is a root of $f(x) = 0$.
$\therefore \qquad R = 0.$
Hence $(x-h)$ is a factor of $f(x)$.
Conversely, if $(x-h)$ is a factor of $f(x)$, then

$$f(x) = (x-h)Q.$$

$\therefore \qquad f(h) = 0.$
This shows that h is a root of $f(x) = 0$.

Theorem: Complex roots of a real polynomial occur in conjugate pairs.
Proof: Let $x = \alpha + i\beta$ be a root of $f(x) = 0$, then $x - (\alpha + i\beta)$ is a factor of $f(x)$. Let us divide $f(x)$ by $[x - (\alpha + i\beta)][x - (\alpha - i\beta)]$ giving quotient Q and $R_1x + R_2$ as remainder.

$$\therefore \qquad f(x) = [x - (\alpha + i\beta)][x - (\alpha - i\beta)]Q + R_1x + R_2.$$

Putting $x = \alpha + i\beta$, we get

$$f(\alpha + i\beta) = R_1(\alpha + i\beta) + R_2.$$

But $\quad f(\alpha + i\beta) = 0$, as $\alpha + i\beta$ is a root of $f(x) = 0$.
$\therefore \quad R_1(\alpha + i\beta) + R_2 = 0$
$\implies R_1\alpha + R_2 = 0$ and $R_1\beta = 0$.
But $\quad \beta \neq 0$, so $R_1 = R_2 = 0$.
$\therefore \quad f(x)$ has two factors $x - (\alpha + i\beta)$ and $x - (\alpha - i\beta)$.
or $\quad f(x) = 0$, has two of the roots $\alpha \pm i\beta$.
Therefore, complex roots of a real polynomial occur in conjugate pairs.

Theorem: Irrational roots of a real polynomial occur in conjugate pairs.
Proof. Let $x = \alpha + \sqrt{\beta}$ (where $\alpha,\ \beta$ are rational such that β is positive and not a perfect square) be a root of $f(x) = 0$, then $x - (\alpha + \sqrt{\beta})$ is a factor of $f(x)$. Let us divide $f(x)$ by $[x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]$ giving quotient Q and $R_1x + R_2$ as remainder.

$$\therefore \qquad f(x) = [x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]Q + R_1x + R_2.$$

Putting $x = \alpha + \sqrt{\beta}$, we get

$$f(\alpha + \sqrt{\beta}) = R_1(\alpha + \sqrt{\beta}) + R_2.$$

But $\quad f(\alpha + \sqrt{\beta}) = 0$, as $\alpha + \sqrt{\beta}$ is a root of $f(x) = 0$.
$\therefore \quad R_1(\alpha + \sqrt{\beta}) + R_2 = 0$
$\implies R_1\alpha + R_2 = 0$ and $R_1\sqrt{\beta} = 0$.
But $\quad \beta \neq 0$, so $R_1 = R_2 = 0$.
$\therefore \quad f(x)$ has two factors $x - (\alpha + \sqrt{\beta})$ and $x - (\alpha - \sqrt{\beta})$.
or $\quad f(x) = 0$, has two of the roots $\alpha \pm \sqrt{\beta}$.
Therefore, irrational roots of a real polynomial occur in conjugate pairs.

Exercise.1:- Given that $1 - 2i$ is a root of $x^4 - 5x^3 + 13x^2 - 19x + 10 = 0$, solve it completely.
Sol. The given root is $1 - 2i$. Since complex roots of a real polynomial occur in conjugate pairs, so $1 + 2i$ is also a root of given equation.
Let $\quad x = 1 \pm 2i$
$\implies \quad x - 1 = \pm 2i$
Squaring both sides, we get

$$\begin{aligned} (x-1)^2 &= (\pm 2i)^2 \\ \implies x^2 + 1 - 2x + 1 &= -4 \\ \implies x^2 + 1 - 2x + 1 + 4 &= 0 \\ \implies x^2 - 2x + 5 &= 0. \end{aligned}$$

Now divide the equation $x^4 - 5x^3 + 13x^2 - 19x + 10$ by $x^2 - 2x + 5$, we get

$$x^4 - 5x^3 + 13x^2 - 19x + 10 = (x^2 - 2x + 5)(x^2 - 3x + 2).$$

The quotient term is $x^2 - 3x + 2$.
$\therefore \quad x^2 - 3x + 2 = 0.$

$\Longrightarrow \quad (x-1)(x-2)=0$
$\Longrightarrow \quad (x-1)=0 \quad$ or $\quad (x-2)=0.$
Thus the other roots are $x=1,2$.

Exercise.2:- Solve the equation $x^4+x^3-25x^2+41x+66=0$, one root is $3+i\sqrt{2}$.
Sol. The given root is $3+i\sqrt{2}$. As complex roots of a real polynomial occur in conjugate pairs, so other root is $3-i\sqrt{2}$.
Let $\quad x=3\pm i\sqrt{2}$
$\Longrightarrow \quad x-3=\pm i\sqrt{2}$
Squaring both sides, we get

$$(x-3)^2=(\pm i\sqrt{2})^2.$$
$$\Longrightarrow x^2+9-6x=-2$$
$$\Longrightarrow x^2+9-6x+2=0$$
$$\Longrightarrow x^2-6x+11=0.$$

Now divide the equation $x^4+x^3-25x^2+41x+66$ by $x^2-6x+11$, we get

$$x^4+x^3-25x^2+41x+66=(x^2-6x+11)(x^2+7x+6).$$

The quotient term is x^2+7x+6.
$\therefore \quad x^2+7x+6=0.$
$\Longrightarrow \quad (x-6)(x+1)=0$
$\Longrightarrow \quad (x-6)=0 \quad$ or $\quad (x+1)=0.$
Thus the other roots are $x=-1,6$.

Lecture 4
Transformation Of Equations

Motivation: In this lecture we will try to form a polynomial equation from the roots of some given equation. But in this case it is necessary to have a definite relation between the roots of two equations. The relation between the roots we will be discussing are same in magnitude but opposite in direction, reciprocal of a given root, a constant multiple of a given root.

Case I.
To find an equation whose roots are equal in magnitude but opposite in sign to those of the given equation $f(x) = 0$.

Let $x = \alpha, \beta, \gamma, \delta, \cdots, \rho$ be the roots of the given equation.
We have to find an equation whose roots are given by

$$x = -\alpha, -\beta, -\gamma, -\delta, \cdots, -\rho.$$

Put $y = -x$ or $x = -y$.
Therefore, the required equation is $f(-y) = 0$.

Case II.
To multiply the roots of a given equation $f(x) = 0$ by a constant m. Let $x = \alpha, \beta, \gamma, \delta, \cdots, \rho$ be the roots of the given equation.
We have to find an equation whose roots are given by

$$y = m\alpha, m\beta, m\gamma, m\delta, \cdots, m\rho.$$

Put $y = -x$ or $x = -y$.
Putting $y = mx$, we get $x = \dfrac{y}{m}$ Therefore, the required equation is $f\left(\dfrac{1}{y}\right) = 0$.

Corollary 1. *If* $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$
Put $x = \dfrac{y}{m}$, *we get*

$$f(\frac{y}{m}) = a_0\frac{y^n}{m^n} + a_1\frac{y^{n-1}}{m^{n-1}} + a_2\frac{y^{n-2}}{m^{n-2}} + \cdots + a_n = 0$$

$$\Longrightarrow \quad f(\frac{y}{m}) = a_0y^n + a_1y^{n-1}m + a_2y^{n-2}m^2 + \cdots + a_nm^n = 0,$$

is the required equation whose roots are multiples of m.

Case III.
To find an equation whose roots are equal to the reciprocal of the roots of given equation $f(x) = 0$.

Let $x = \alpha, \beta, \gamma, \delta, \cdots, \rho$ be the roots of the given equation.
We have to find an equation whose roots are given by

$$y = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \cdots, \frac{1}{\rho}.$$

Thus setting $y = \dfrac{1}{x}$ or $x = \dfrac{1}{y}$.
Therefore, the required equation is $f\left(\dfrac{1}{y}\right) = 0$.

To Diminish The Roots Of An Equation By a Given Quantity h.
Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ whose roots are $x = \alpha, \beta, \gamma, \delta, \cdots, \rho$. We have to find that equation whose roots are $\alpha - h, \beta - h, \gamma - h, \delta - h, \cdots, \rho - h$.

Therefore, the new roots are $y = \alpha - h, \beta - h, \gamma - h, \delta - h, \cdots, \rho - h$.
Hence $y = x - h$, gives $x = y + h$. Substituting the value of x in the given equation, we get

$$f(y+h) = a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \cdots + a_{n-1}(y+h) + a_n = 0,$$

is the required equation.

Practical Method

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \tag{0.1}$$

Now if x is a root of this equation, we want an equation whose root is $x - h$.
Let $y = x - h \implies \quad x = y + h$.
Substituting this value of x in (**??**), we get

$$f(y+h) = a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \cdots + a_{n-1}(y+h) + a_n = 0. \tag{0.2}$$

Let equation (**??**) take the form

$$f(y+h) = A_0y^n + A_1y^{n-1} + A_2y^{n-2} + \cdots + A_{n-1}y + A_n = 0, \tag{0.3}$$

where $A_0, A_1, \cdots, A_{n-1}, A_n$ are to be determined.
Now changing y back into $x - h$ in (**??**), we get

$$f(x) = A_0(x-h)^n + A_1(x-h)^{n-1} + A_2(x-h)^{n-2} + \cdots + A_{n-1}(x-h) + A_n = 0.$$

Dividing above polynomial equation by $x - h$, the quotient will be

$$A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + A_2(x-h)^{n-3} + \cdots + A_{n-2}(x-h) + A_{n-1} = 0.$$

and the remainder is A_n.
Again dividing $A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + A_2(x-h)^{n-3} + \cdots + A_{n-2}(x-h) + A_{n-1} = 0$ by $x - h$, the quotient will be

$$A_0(x-h)^{n-2} + A_1(x-h)^{n-3} + A_2(x-h)^{n-4} + \cdots + A_{n-3}(x-h) + A_{n-2} = 0.$$

and the remainder is A_{n-1}.
Repeating this process of division, we get the coefficients $A_n, A_{n-1}, A_{n-2} \cdots$, as successive remainders. Thus we have the following:

1. Divide $f(x)$ by $x - h$, the remainder will give A_n.
2. Divide the quotient again by $x - h$, the remainder will give A_{n-1}.

By repeating the process, we get $A_{n-2}, A_{n-3} \cdots$, etc. as remainders.

Removal of Terms

Let the n^{th} degree equation be

$$f(x) = A_0(x-h)^n + A_1(x-h)^{n-1} + A_2(x-h)^{n-2} + \cdots + A_{n-1}(x-h) + A_n = 0.$$

Let us diminish its roots by h, so that

$$y = x - h \quad \text{or} \quad x = y + h.$$

$\therefore$ The transformed equation becomes

$$f(y+h) = a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \cdots + a_{n-1}(y+h) + a_n = 0.$$

Expanding the R.H.S. by Binomial Theorem and simplifying it, we get

$$f(y+h) = a_0y^n + (nha_0 + a_1)y^{n-1} + \left(\frac{n(n-1)}{2}a_0h^2 + (n-1)a_1h + a_2\right)y^{n-2} + \cdots + = 0.$$

Now in order to remove the second term, we put

$$nha_0 + a_1 \implies \quad h = \frac{-\dfrac{a_1}{a_0}}{n}$$

$$i,e. \qquad h = \frac{\text{Sum of roots}}{\text{Degree of the equation}}.$$

Thus by decreasing the roots of a given equation, by a quantity h, where

$$h = \frac{\text{Sum of roots}}{\text{Degree of the equation}},$$

we can remove the second term of the given equation.
Similarly for removing the third term of the equation, we put

$$\frac{n(n-1)}{2}a_0h^2 + (n-1)a_1h + a_2 = 0,$$

this will give us two values of h and therefore, two transformed equations and so on.

Removal Of Second Term Of The Cubic Equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

In order to remove the second term, we have to diminish the roots by

$$h = \frac{-\dfrac{a_1}{a_0}}{n}$$
$$= \frac{\text{Sum of roots}}{\text{Degree of the equation}}$$

$\therefore \qquad y = x - h = x + \dfrac{a_1}{a_0} \quad \implies \qquad x = y - \dfrac{a_1}{a_0}$

Substitute this value of x in the given cubic equation, we get

$$a_0\left(y - \frac{a_1}{a_0}\right)^3 + 3a_1\left(y - \frac{a_1}{a_0}\right)^2 + 3a_2\left(y - \frac{a_1}{a_0}\right) + a_3 = 0.$$

or
$$a_0y^3 + \frac{3y}{a_0}(a_0a_1 - a_1^2) + \frac{(2a_1^3 - 3a_0a_1a_2 + a_3a_0^2)}{a_0^2} = 0.$$

Put $\qquad a_0y = z, \text{or} y = \dfrac{z}{a_0}.$

$\therefore \qquad$ This equation reduced to

$$z^3 + 3(a_0a_1 - a_1^2)z + (2a_1^3 - 3a_0a_1a_2 + a_3a_0^2) = 0.$$

or $\qquad z^3 + 3Hz + G = 0.$

where $\qquad H = a_0a_1 - a_1^2$ and $G = 2a_1^3 - 3a_0a_1a_2 + a_3a_0^2$, is the required transformed equation whose second term is missing.

Example 1. *Find the equation whose roots are opposite in sign to those of the equation*

$$x^4 - 2x^3 + 6x^2 - x + 7 = 0.$$

Sol. Let $x=\alpha,\beta,\gamma,\delta$ be the roots of the given equation. We should have $y=-\alpha,-\beta,-\gamma,-\delta$. Putting $y=-x$, or $x=-y$, we get

$$(-y)^4-2(-y)^3+6(-y)^2-(-y)+7=0$$

$$y^4+2y^3+6y^2-2y+7=0.$$

Example 2. *Multiply the roots of the following equation in such a way that the coefficients of the highest term is unity and all other coefficients are integral:*

$$\frac{1}{2}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{1}{48}=0.$$

Sol. Let us multiply the roots of the given equation by m. Therefore, the transformed equation is

$$\frac{1}{2}y^4-\frac{1}{3}my^3+\frac{1}{2}m^2y^2-\frac{1}{48}m^4=0.$$

$\therefore$ we choose $m=6$.

$$\frac{1}{2}y^4-\frac{1}{3}6y^3+\frac{1}{2}36y^2-\frac{1}{48}1296=0.$$

$$\implies y^4-4y^3+36y^2-54=0,$$

is the required equation.

Removal Of Second Term Of The Bi-quadratic Equation

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0.$$

In order to remove the second term, we have to diminish the roots by

$$h=\frac{-\frac{a_1}{a_0}}{n}$$

$$=\frac{\text{Sum of roots}}{\text{Degree of the equation}}$$

$\therefore$ $y=x-h=x+\dfrac{a_1}{a_0}$ $\implies$ $x=y-\dfrac{a_1}{a_0}$

Therefore the transformed equation becomes

$$a_0\frac{(a_0y-a_1)^4}{a_0^4}+4a_1\frac{(a_0y-a_1)^3}{a_0^3}+6a_2\frac{(a_0y-a_1)^2}{a_0^2}+4a_3\frac{(a_0y-a_1)}{a_0}+a_4=0.$$

This gives

$$a_0^4y^4+6a_0^2y^2(a_0a_2-a_1^2)+4a_0y(2a_1^3+3a_0a_1a_2+a_0^2a_3)+(6a_0a_2a_1^2-3a_1^4-4a_0^2a_1a_3+a_0^3a_4)=0.$$

or $a_0^4y^4+6a_0^2y^2(a_0a_2-a_1^2)+4a_0y(2a_1^3+3a_0a_1a_2+a_0^2a_3)+[-3(a_0a_2-a_1^4)+a_0^2(3a_2^2-4a_1a_3+a_0a_4)]0.$

Put $y=\dfrac{z}{a_0}$, $H=a_0a_2-a_1^2$, $G=2a_1^3-3a_0a_1a_2+a_0^2a_3$ and $I=3a_2^2-4a_1a_3+a_0a_4$,

we get the transformed equation as

$$z^4+6Hz^3+4Gz^2+a_0^2I+3H^2=0.$$

Example 3. *Diminish the roots of*

$$x^4+3x^2+4x+9=0$$

by $x=5$.

Sol. Here $h = 5$, therefore we get

$$\begin{array}{r|rrrrr} 5 & 1 & 0 & 3 & 4 & 9 \\ & & 5 & 25 & 140 & 720 \\ \hline & 1 & 5 & 28 & 144 & |\underline{729} \\ & & 5 & 50 & 390 & \\ \hline & 1 & 10 & 78 & |\underline{534} & \\ & & 5 & 75 & & \\ \hline & 1 & 10 & |\underline{153} & & \\ & & 5 & & & \\ \hline & 1 & 20 & & & \end{array}$$

$\therefore \quad y^4 + 20y^3 + 153y^2 + 534y + 729 = 0.$

Example 4. *Remove the second term from the equation*

$$x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$$

Sol. We have

$$h = \frac{\text{Sum of roots}}{\text{Degree of the equation}} = -\frac{4}{4} = -1.$$

$$\begin{array}{r|rrrrr} -1 & 1 & 4 & 2 & -4 & -2 \\ & & -1 & -3 & 1 & 3 \\ \hline & 1 & 3 & -1 & -3 & |\underline{1} \\ & & -1 & -2 & 3 & \\ \hline & 1 & 2 & -3 & |\underline{0} & \\ & & -1 & -1 & & \\ \hline & 1 & 1 & |\underline{-4} & & \\ & & -1 & & & \\ \hline & 1 & |\underline{0} & & & \end{array}$$