Rolle's Theorem and Lagrange's Mean value Theorem
Mushtaq A. Bhat
SAM Degree College Budgam

Rolle's Theorem If a function f is continuous in a closed interval $[a, b]$, derivable in open interval (a, b) such that $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. f is continuous on $[a, b]$ so it has maximum M and minimum m in this interval. Let c and d be numbers in $[a, b]$ such that $f(c) = M$ and $f(d) = m$. We consider two cases (i) $M = m$ and (ii) $M \neq m$.
Case (i) If $M = m$, the function is constant and so derivative is zero and hence result follows in this case.
Case (ii) If $M \neq m$, then at least one of these must be different from equal value of $f(a), f(b)$.
Let $M = f(c)$ be different from them i.e., $c \in (a, b)$ so $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists.
Also M being maximum value of f gives $f(c + h) \leq f(c)$for all h which gives

$$\frac{f(c+h)-f(c)}{h} \leq 0 \quad for \quad h > 0$$
$$\frac{f(c+h)-f(c)}{h} \leq 0 \quad for \quad h < 0$$

Therefore,

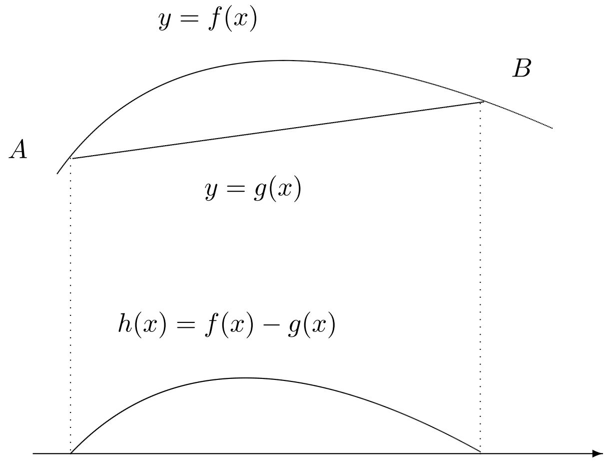
$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0 \Rightarrow f'(c) \leq 0$$

Also,

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0 \Rightarrow f'(c) \geq 0$$

Combining last two inequalities gives $f'(c) = 0$.

Example Verify Rolle's Theorem for $f(x) = \frac{\sin x}{e^x}$ in the interval $[0, \pi]$.
Solution. As $\sin x$ and e^x are continuous on $[0, \pi]$ and $e^x \neq 0$ on $[0, \pi]$, therefore $\frac{\sin x}{e^x}$ is continuous. Also, $f'(x) = \frac{\cos x - \sin x}{e^x}$ which exists on $(0, \pi)$. Further $f(0) = f(\pi) = 0$. Thus all three conditions of Rolle's are satisfied by f. So, by Rolle's Theorem there exists a point $c \in (0, \pi)$ such that $f'(c) = 0$ i.e., $\frac{\sin c - \cos c}{e^c} = 0$. Simplifying we get $c = \frac{\pi}{4}$.



Lagrange's Mean Value Theorem. Let a function f be continuous on $[a, b]$, derivable in (a, b). Then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. We picture the graph of f as a curve in the plane and draw a line through the points $A(a, f(a))$ and$B(b, f(b))$ (see above figure). The line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \ \ (point - slope \ \ equation)$$

The vertical difference between the graphs of f and g at x is

$$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

The function h satisfies the hypotheses of Rolle's theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$, because the graphs of f and g both pass through A and B. Therefore, $h' = 0$ at some point c in (a, b). Differentiate $h(x)$ both sides with respect x and then set $x = c$, we have

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so that

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \ \ or \ \ 0 = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which gives

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Alternative form of Lagrange's Mean Value Theorem Suppose f is continuous on $[a, a+h]$, derivable on $(a, a+h)$, then there exists a number $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a+\theta h).$$