Differential Calculus
Unit-1
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We begin this chapter with $\epsilon - \delta$ definition of limit of a function with real domain and range. We assume students are familiar with limits and continuity taught in 12th class. Recall for two real numbers a and b, $|a-b|$ is distance between a and b, so $|a|$ is distance of a from origin. Also for any real number a, $\sqrt{a^2} = |a|$.
Definition A Formal Definition of limit:
Let $f(x)$ be a function defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L$$

if, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Example Show that $\lim_{x \to 1}(5x - 3) = 2$.
Solution. Set $x_0 = 1$, $f(x) = 5x - 3$ and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq x_0 = 1$ and x is within distance δ of $x_0 = 1$, that is , if $0 < |x - 1| < \delta$, then $f(x)$ is within distance ϵ of $L = 2$, that is $|f(x) - 2| < \epsilon$.
We find delta by working backwards from the ϵ-inequality.

$$|(5x - 3) - 2| = |5x - 5| < \epsilon \Rightarrow 5|x - 1| < \epsilon \Rightarrow |x - 1| < \frac{\epsilon}{5}.$$

Thus we can take $\delta = \frac{\epsilon}{5}$. If $0 < |x - 1| < \delta = \frac{\epsilon}{5}$, then $|(5x - 3) - 2| < \epsilon$.

Example Verify $(a) \lim_{x \to x_0} x = x_0$ $(b) \lim_{x \to x_0} k = k$, where k is constant.
Solution. (a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number. This proves that $\lim_{x \to x_0} x = x_0$.
(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all

$$0 < |x - x_0| < \delta \Rightarrow |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold. This proves that $\lim_{x \to x_0} k = k$.

How to find a δ for a given f, L, x_0 and $\epsilon > 0$ algebraically.
The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

can be accomplished in two steps.
Step1. Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) about x_0 on which the inequality holds for all $x \neq x_0$.
Step2. Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centred at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Example For the limit $\lim_{x \to 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$, i.e., find a $\delta > 0$ such that for all x

$$0 < |x - 5| < \delta \Rightarrow |\sqrt{x - 1} - 2| < 1.$$

Solution. First we solve the inequality $|\sqrt{x-1}-2|<1$ to find an interval about $x_0=5$ on which the inequality holds for all $x \neq x_0$.
$|\sqrt{x-1}-2|<1| \Rightarrow -1<\sqrt{x-1}-2<1 \Rightarrow 1<\sqrt{x-1}<3 \Rightarrow 2<x<10$.
The inequality holds for all x in the open interval $(2,10)$. So, it holds for all $x \neq 5$ in this interval as well. The distance from 5 to the nearest endpoint of $(2,10)$ is 3. If we take $\delta=3$ or any smaller positive number, then the inequality $0<|x-5|<\delta$ will automatically place x between 2 and 10 to make $|\sqrt{x-1}-2|<1$. i.e.,

$$0<|x-5|<3 \Rightarrow |\sqrt{x-1}-2|<1$$

.

Example Prove that $\lim_{x\to 2} f(x)=4$ if

$$f(x)=\begin{cases} x^2, & x\neq 2 \\ 1, & x=2. \end{cases}$$

Solution. For $x\neq x_0=2$, we have $f(x)=x^2$. Let $\epsilon>0$ be given.
Now $|f(x)-4|<\epsilon \Rightarrow |x^2-4|<\epsilon \Rightarrow 4-\epsilon<x^2<4+\epsilon \Rightarrow \sqrt{4-\epsilon}<|x|<\sqrt{4+\epsilon}$ or $\sqrt{4-\epsilon}<x<\sqrt{4+\epsilon}$ assuming $\epsilon<4$.
Take $\delta=\min\{2-\sqrt{4-\epsilon}, \sqrt{4+\epsilon}-2\}$. If δ has this or any smaller positive value, the inequality $0<|x-2|<\delta$ will automatically place x between $\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}$ to make $|f(x)-4|<\epsilon$. Therefore for all x

$$0<|x-2|<\delta \Rightarrow |f(x)-4|<\epsilon.$$

Theorem Properties of limits: Let $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$. Then
(i). $\lim_{x\to c}[f(x)+g(x)]=\lim_{x\to c} f(x)+\lim_{x\to c} g(x)=L+M$.
(ii). $\lim_{x\to c}[f(x)-g(x)]=\lim_{x\to c} f(x)-\lim_{x\to c} g(x)=L-M$.
(iii). $\lim_{x\to c} f(x).g(x)=\lim_{x\to c} f(x).\lim_{x\to c} g(x)=L.M$
(iv). $\lim_{x\to c} kf(x)=k\lim_{x\to c} f(x)=kL$.
(v). $\lim_{x\to c} \frac{f(x)}{g(x)}=\frac{L}{M},\ M\neq 0$.
(vi). If m and n are integers, then $\lim_{x\to c}[f(x)]^{\frac{m}{n}}=L^{\frac{m}{n}}$, provided $L^{\frac{m}{n}}$ is a real number.
Proof (i) Let $\epsilon>0$ be given. We want to find a positive number δ such that for all x

$$0<|x-c|<\delta \Rightarrow |f(x)+g(x)-(L+M)|<\epsilon$$

Regrouping terms, we get

$$|f(x)+g(x)-(L+M)|=|(f(x)-L)+(g(x)-M)|\leq |f(x)-L|+|g(x)-M|$$

by triangle inequality $|a+b|\leq |a|+|b|$.
Since $\lim_{x\to c} f(x)=L$, there exists a number $\delta_1>0$ such that for all x

$$0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\frac{\epsilon}{2}$$

Similarly, since $\lim_{x\to c} g(x)=M$, there exists a number $\delta_2>0$ such that for all x

$$0<|x-c|<\delta_2 \Rightarrow |g(x)-L|<\frac{\epsilon}{2}$$

Let $\delta=\min\{\delta_1,\delta_2\}$. If $0<|x-c|<\delta$, then $|x-c|<\delta_1$ so that $|f(x)-L|<\frac{\epsilon}{2}$ and $|x-c|<\delta_2$ so that $|g(x)-L|<\frac{\epsilon}{2}$.
Therefore, $|f(x)+g(x)-(L+M)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$.
This shows that $\lim_{x\to c}[f(x)+g(x)]=L+M$.
Try to prove other parts yourself.
Sandwich Theorem or Sequeeze principle. Suppose that $g(x)\leq f(x)\leq h(x)$ for all x in some open interval containing c except possible at $x=c$ itself. Suppose also that $\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L$, then $\lim_{x\to c} f(x)=L$.

Prove Sandwich Theorem using $\epsilon - \delta$ definition.
Example Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$. Find $\lim_{x\to 0} u(x)$.
Solution. Since $\lim_{x\to 0}(1 - \frac{x^2}{4}) = 1$ and $\lim_{x\to 0}(1 + \frac{x^2}{2}) = 1$.
By Sandwich Theorem $\lim_{x\to 0} u(x) = 1$

Continuity and types of discontinuities. A function $f(x)$ is said to be continuous at a point a in domain D if

$$\lim_{x\to a} f(x) = f(a)$$

in terms of $\epsilon - \delta$, given $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

If f is continuous at each point in D, we say f is continuous in D. Moreover, f is said to discontinuous at a point a if f is not continuous at a. The discontinuity of a function f at a point a of its domain can arise in either of the following two ways.
(i). $\lim_{x\to a} f(x)$ exists but is different from the value of $f(a)$.
(ii). $\lim_{x\to a} f(x)$ does not exist.
Example. Show that the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

is continuous on R.
Solution. Clearly f is continuous for $x \in R - \{0\}$. Here $f(0) = 0$. For $x \neq 0$, $f(x) = x\sin(\frac{1}{x})$
We have $|f(x) - f(0)| = |x\sin(\frac{1}{x})| \leq |x| < \epsilon$ whenever $|x| = |x - 0| < \delta = \epsilon$. Thus f defined above is continuous at $x = 0$ and hence on R.
Try to solve above problem using Sandwich Theorem and directly without $\epsilon - \delta$ definition.

Exercise. Examine the continuity of the functions without $\epsilon - \delta$ approach.
(i). $f(x) = |x| + |x - 1| + |x - 1|$. (ii). $f(x) = \sin(\frac{1}{x})$ *when* $x \neq 0$ *and* $f(0) = 0$.
(iii). $f(x) = \frac{x^2-1}{x-1}$ *if* $x \neq 1$ *and* $f(1) = 3$. (iv). $f(x) = \exp(\frac{1}{x})$ *for* $x \neq 0$ *and* $f(0) = 0$.

Types of discontinuity. Note that when we discuss discontinuity points we are discussing only points at which the function is defined. (Some calculus texts might call x_0 a point of discontinuity even if $f(x_0)$ fails to be defined. This is not our usage here.)

Removable Discontinuities. We can separate these cases into situations of increasing severity. The weakest possibility is that $\lim_{x\to x_0} f(x)$ does indeed exist but fails to equal $f(x_0)$. We call this a removable discontinuity of f. The word "removable" suggests that were we merely to assign a new value to $f(x_0)$ we would no longer have a discontinuity.

Jump Discontinuity. A little more serious case of discontinuity occurs if the limit $\lim_{x\to x_0} f(x)$ does not exist, but it fails to exist only because

$$\lim_{x\to x_0^-} f(x) \quad and \quad \lim_{x\to x_0^+} f(x)$$

the two one-sided limits, exist but disagree. In that case, no matter what value $f(x_0)$ assumes, this is a point of discontinuity. We call this a jump discontinuity of f. The difference between the two limits

$$\lim_{x\to x_0^+} f(x) - \lim_{x\to x_0^-} f(x)$$

is a measure of the "size" of the discontinuity and is called the jump. Some authors call this type of discontinuity as " **discontinuity of first kind**".

Essential Discontinuities. Finally, the most intractable kind of discontinuity would be the situation in which $lim_{x \to x_0} f(x)$ does not exist, and at least one of the two right-hand and left-hand limits (perhaps both) $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ also does not exist. Again, no matter what value $f(x_0)$ assumes, this is a point of discontinuity. We call this an essential discontinuity of f.
We note that some authors say that f has **discontinuity of kind second** in case both the right and left hand limits does not exist at x_0 and in case one of the two limits exists, then f has a **mixed discontinuity**.
Example(i). Consider the function

$$f(x) = \begin{cases} x \sin(\frac{1}{x}), & x \neq 0 \\ 2, & x = 0. \end{cases}$$

As $\lim_{x \to 0} f(x) = 0$. Therefore $x = 0$ is a removable discontinuity of f. If we redefine f at $x = 0$ as $f(0) = 0$, then f becomes continuous.
(ii). Consider the function f defined as

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

Then $\lim_{x \to 0^-} f(x) = 1$ but $\lim_{x \to 0^+} f(x) = 0$
So $x = 0$ is a jump discontinuity or discontinuity of first kind.
Consider the function f defined as

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ does not exist. So, f has discontinuity of second kind or essential discontinuity at $x = 0$.
(iv). Consider the function f defined as

$$f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Then $\lim_{x \to 0^-} f(x)$ does not exist but $\lim_{x \to 0^+} f(x) = 1$. So, f has mixed discontinuity or essential discontinuity at $x = 0$.
Exercises (i). Give an example of a function continuous at 0 but with an essential discontinuity at each other point.
(ii). Give an example of a function f with a jump discontinuity and yet $(f)^2$ is continuous everywhere.
(iii). Give an example of a function f with an essential discontinuity everywhere and yet $(f)^2$ is continuous everywhere.
(iv). Let $x_0 \in \mathbb{R}$. Following are four $\epsilon - \delta$ conditions on a function $f : \mathbb{R} \to \mathbb{R}$. Which, if any, of these conditions imply continuity of f at x_0? Which, if any, are implied by continuity at x_0?
(a). For every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
(b). For every $\epsilon > 0$ there exists $\delta > 0$ such that if $|f(x) - f(x_0)| < \delta$, then $|x - x_0| < \epsilon$.
(c). For every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \epsilon$, then $|f(x) - f(x_0)| < \delta$.
(d). For every $\epsilon > 0$ there exists $\delta > 0$ such that if $|f(x) - f(x_0)| < \epsilon$, then $|x - x_0| < \delta$.
(v). Let $x_0 \in \mathbb{R}$. Following are four $\epsilon - \delta$ conditions on a function $f : \mathbb{R} \to \mathbb{R}$. Which, if any, of these conditions imply continuity of f at x_0? Which, if any, are implied by continuity at x_0?
(a). There exists $\epsilon > 0$ such that for each $\delta > 0$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
(b). There exists $\epsilon > 0$ such that for each $\delta > 0$, if $|f(x) - f(x_0)| < \delta$ then $|x - x_0| < \epsilon$.
(c). There exists $\epsilon > 0$ such that for each $\delta > 0$, if $|x - x_0| < \epsilon$, then $|f(x) - f(x_0)| < \delta$.
(d). There exists $\epsilon > 0$ such that for each $\delta > 0$ if $|f(x) - f(x_0)| < \epsilon$ then $|x - x_0| < \delta$.

(v). Suppose that f is a function defined on the real line with the property that $f(x + y) = f(x) + f(y)$ for all x, y. Suppose that f is continuous at 0. Show that f must be continuous everywhere.

(vi). Suppose that f is a function defined on the real line with the property that $f(x+y) = f(x)f(y)$ for all x, y. Suppose that f is continuous at 0. Show that f must be continuous everywhere.

Properties of continuous functions on a closed interval.

Boundedness A set $S \subset \mathbb{R}$ is said to be **bounded above** if there exits a positive real number M such that $x \leq M$ for all $x \in S$. If M is an upper bound of S, then clearly any number greater than M is also an upper bound. We say M is least upper bound (or Supremum) of S if no number less than M is upper bound of S. More precisely, if for every $\epsilon > 0$ there exists $y \in S$ such that $y > M - \epsilon$.
Similarly, we say $S \subset R$ is **bounded below** if there exists a real number m such that $x \geq m$ for all $x \in S$. Clearly any real number less than m is a lower bound of S. We say m is greatest lower bound (or infimum) of S if no number greater than m is a lower bound. More precisely, if for every $\epsilon > 0$ there is an element $z \in S$ such that $z < m + \epsilon$.
Consider an open interval $I = (0,1)$. It is clear 0 is infimum and 1 is supremum but neither of them belongs to I. For interval $(0,1]$ supremum is 1 which belongs to set whereas infimum is 0 which does not belongs to set.

Theorem A Let f be continuous function on closed interval $[a,b]$. Then the interval can always be divided up into a finite number of subintervals such that given $\epsilon > 0$

$$|f(x_1) - f(x_2)| < \epsilon$$

where x_1 and x_2 are any two points of the same sub-interval.
Theorem B Let f be continuous on a closed interval $[a,b]$. Then it is bounded therein.
Proof. As f is continuous on $[a,b]$, therefore by Theorem A, given $\epsilon > 0$, we can divide $[a,b]$ into a finite number of subintervals $[a = a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n = b]$ such that

$$|f(x_1) - f(x_2)| < \epsilon, \quad for \quad x_1 \ \ and \ \ x_2 \ \ in \ \ same \ \ subinterval.$$

If $x \in [a, a_1]$ then $|f(x) - f(a)| < \epsilon$.
Also $|f(x)| = |f(x) - f(a) + f(a)| \leq |f(x) - f(a)| + |f(a)| < \epsilon + |f(a)|$
When $x = a_1, |f(a_1)| < |f(a)| + \epsilon$.
If now $x \in [a_1, a_2]$, then $|f(x) - f(a_1)| < \epsilon$ and $|f(x)| = |f(x) - f(a_1) + f(a_1)| \leq |f(x) - f(a_1)| + |f(a_1)| < \epsilon + |f(a_1)| < \epsilon + |f(a)| + \epsilon = |f(a)| + 2\epsilon$.
Thus when $x = a_2$, $|f(a_2)| < |f(a)| + 2\epsilon$. Similarly, one can show $|f(x)| < |f(a)| + n\epsilon \ \ for \ \ all \ \ x \in [a,b]$. As n is finite, we see f is bounded on [a,b].

Remark Theorem B does not hold in general if interval is not closed e.g., consider function f defined by $f(x) = \frac{1}{x} \ \ on \ \ interval \ \ (0,1)$. Clearly f is continuous on $(0,1)$ but not bounded.

Theorem C If a function f is continuous on a closed interval $[a,b]$, then it attains its supremum and infimum at least once in $[a,b]$.
Proof. Let f be continuous on $[a,b]$. Then by Theorem B, f is bounded on $[a,b]$ and so both infimum and supremum make sense. Let $l = sup_{[a,b]} f$. Then $f(x) \leq l \ \ for \ \ all \ \ x \in [a,b]$.
If possible, suppose f does not attain l on $[a,b]$ i.e., $f(x) \neq l$ for all $x \in [a,b]$, this implies $l - f(x) \neq 0$ on $[a,b]$. As f is continuous on $[a,b]$ and constant is continuous on $\mathbb{R}$ and in particular on $[a,b]$, therefore both $l - f(x)$ and $\frac{1}{l - f(x)}$ are continuous on $[a,b]$. By Theorem B, the function $\frac{1}{l-f(x)}$ is bounded. Hence there exist a real number $M > 0$ such that

$$\frac{1}{l - f(x)} \leq M \Rightarrow l - f(x) \geq \frac{1}{M} \Rightarrow f(x) \leq l - \frac{1}{M} < l$$

Thus $l - \frac{1}{M}$ is an upper bound of f on $[a,b]$ which is less than $sup_{[a,b]} f$, a contradiction. Hence there exists some $c \in [a,b]$ such that $f(x) = l$ i.e., f attain its supremum on $[a,b]$. Similarly, one can prove that f attains infimum on $[a,b]$. This completes the proof.

Remark. Theorem C does not hold in general if interval is not closed e.g., consider a function f defined on $(0,1)$ as $f(x) = 2x \ \ on \ \ interval \ \ (0,1)$. Clearly, infimum and supremum of f are 0 and 2 respectively and none of them is attained.

Order completeness property of real numbers. Every non-empty subset of real numbers which is bounded above has least upper bound (or supremum) in $\mathbb{R}$.

Theorem D Let f be continuous on $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point $c \in [a, b]$ such that $f(c) = 0$.

Proof. With out loss of generality, we can assume $f(a) > 0$ and $f(b) < 0$. Define a set $S \subset [a, b]$ as follows

$$S = \{x \in [a, b] : f(x) > 0\}.$$

As $f(a) > 0$, therefore $a \in S$ which implies $S \neq \emptyset$. Also S is bounded by a and b. By Order completeness property of real numbers, S has **the** supremum, say c. We note $a \leq c \leq b$. We next show $c \neq a, b$ and $f(c) = 0$.

We first show $c \neq a$. As $f(a) > 0$ and f is continuous on $[a, b]$, so there exits a $\delta > 0$ such that $f(x) > 0$ for all $x \in [a, a + \delta)$ (why? try to prove yourself by using definition of continuity). Therefore, $[a, a + \delta) \subset S$ and so $c \geq a + \delta$ (by definition of c) which implies $c \neq a$.

Similarly, one can prove that $c \neq b$.

We next prove $f(c)$ is neither positive nor negative and so only possibility will be $f(c) = 0$.

If $f(c) > 0$, then there exists a $\delta_1 > 0$ such that $f(x) > 0$ for all $x \in (c - \delta_1, c + \delta_1)$. This implies $(c - \delta_1, c + \delta_1) \subset S$. Now, let $0 < \delta' < \delta_1$. Then $c + \delta' \in (c - \delta_1, c + \delta_1)$. This implies that an element $c + \delta'$ of S is greater than $c = \sup S$, a contradiction. Hence $f(c)$ cannot be positive.

Next, assume $f(c) < 0$, then there exists a $\delta_2 > 0$ such that $f(x) < 0$ for all $x \in (c - \delta_2, c + \delta_2)$. At the same time, as c is supremum of S, then by definition of *supremum*, for above $\delta_2 > 0$ there exists an element c_1 of S where $c - \delta_2 < c_1 < c$ such that $f(c_1) > 0$. But $f(c_1) < 0$, this is a contradiction. Hence $f(c) = 0$.

Intermediate value theorem. Let f be continuous on $[a, b]$ and $f(a) \neq f(b)$. Then f assumes every value between $f(a)$ and $f(b)$ at least once.

Proof. We assume $f(a) < f(b)$. Let M be any number between $f(a)$ and $f(b)$, we need to show that there exists a number $c \in (a, b)$ such that $f(c) = M$.

Define a function ϕ on $[a, b]$ as

$$\phi(x) = f(x) - M$$

Now f is continuous on $[a, b]$ and M being constant function is continuous on $\mathbb{R}$ and in particular on $[a, b]$, therefore ϕ is continuous on $[a, b]$.

Now

$$\phi(a) = f(a) - M < 0 \ \ and \ \ \phi(b) = f(b) - M > 0 \ \ because \ \ f(a) < M < f(b).$$

So, by Theorem D, there exists a point $c \in (a, b)$ such that $\phi(c) = 0 \Rightarrow f(c) - M = 0 \Rightarrow f(c) = M$. This completes proof.

There are various ways to write Intermediate value Theorem e.g., one more way is

Alternative statement of IVP Let f be continuous on $[a, b]$ and let $c \in \mathbb{R}$. If for every $x \in [a, b]$, $f(x) \neq c$, then either $f(x) > c$ for all $x \in [a, b]$ or $f(x) < c$ for all $x \in [a, b]$.

Remark Intermediate value property (or Darboux property read as Darbou)(IVP) is a necessary condition satisfied by continuous functions on a closed interval. There are discontinuous functions which have IVP as can be seen in the following example.

Consider a function f defined on $[-1, 1]$ as

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It is clear that f has the IVP. In particular, f assumes every value in the interval $[-1, 1]$ infinitely often in every neighbourhood of $x = 0$.

Corollary. the image of a closed interval under a continuous function is a closed interval.

Exercise Prove or disprove following statements. (i). If f and g have IVP on $[a, b]$ the so have $f.g$ and $f + g$.
(ii). f has IVP implies f is bounded.

Differentiability The derivative of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Examples show that the function

$$f(x) = \begin{cases} 2 + x, & x \geq 0 \\ 2 - x, & x < 0. \end{cases}$$

is not derivable at the point $x = 0$.
Solution. $Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h} = \lim_{h\to 0} \frac{[2-(-h)]-(2+0)}{-h} = \lim_{h\to 0} \frac{h}{-h} = \lim_{h\to 0}(-1) = -1$
$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h} = \lim_{h\to 0} \frac{[2+h]-(2+0)}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$
Thus $Lf'(0) \neq Rf'(0)$. Therefore, $f'(0)$ does not exist and so f is not derivable at $x = 0$.

Example Show that the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

is continuous and differentiable on R.
Solution. Clearly f is continuous and differentiable on $R - \{0\}$. So, we check for continuity and differentiability at $x = 0$.

$$\begin{aligned} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \sin(\frac{1}{x}) &= \lim_{h\to 0} (0+h)^2 \sin\frac{1}{0+h} \\ &= \lim_{h\to 0} h^2 \sin\frac{1}{h} = 0 \times \ a \ \ finite \ \ quantity \ \ between \ \ -1 \ \ and \ \ 1 \\ &= 0. \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin(\frac{1}{x}) &= \lim_{h\to 0} (0-h)^2 \sin\frac{1}{0-h} \\ &= \lim_{h\to 0} -h^2 \sin\frac{1}{h} = 0 \times \ a \ \ finite \ \ quantity \ \ between \ \ -1 \ \ and \ \ 1 \\ &= 0. \end{aligned}$$

As two sided limits are equal at $x = 0$, therefore f is continuous at $x = 0$.

We have

$$\begin{aligned} f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{(0+h)^2 \sin\frac{1}{0+h} - 0}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h \sin\frac{1}{h} h = \lim_{h\to 0} h \sin\frac{1}{h} \\ &= 0 \times \ a \ \ finite \ \ quantity \ \ between \ \ -1 \ \ and \ \ 1 = 0. \end{aligned}$$

In above example is f' continuous at $x = 0$.

Exercises Examine following functions for differentiability at indicated point(s).
(i).

$$f(x) = \begin{cases} (x-a)\sin(\frac{1}{x-a}), & x \neq a \\ 0, & x = a. \end{cases}$$

at $x = a$.
(ii). $f(x) = |x-2|$ at $x = 2$.
(iii).

$$f(x) = \begin{cases} x^2 - 1, & x \geq 1 \\ 1 - x, & x < 1. \end{cases}$$

at $x = 1$.
(iv). $f(x) = x|x|$ on $\mathbb{R}$
(v). Prove or disprove "derivative of a differentiable function is continuous."

Theorem E. Let f be differentiable on $[a, b]$ and $f'(a)$ and $f'(b)$ are of opposite signs, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.
Proof. We assume without loss of generality that $f'(a) < 0$ and $f'(b) > 0$. Since $f'(a) < 0$, therefore there exits a $\delta_1 > 0$ such that

$$f(x) < f(a) \quad for \quad all \quad x \in (a, a + \delta_1). \tag{1}$$

Again, as $f'(b) > 0$, there exists a positive number δ_2 such that

$$f(x) < f(b) \quad for \quad all \quad x \in (b - \delta_2, b). \tag{2}$$

Also f being differentiable on $[a, b]$ is therefore continuous there and so by Theorem C f attains both supremum and infimum. Let m be its infimum and M be its supremum. Let $c \in [a, b]$ such that $f(c) = m$ It is clear from (1) and (2) that $c \neq a, b$, otherwise it contradicts to definition of infimum and so $c \in (a, b)$. Next, we show that $f'(c) = 0$. If possible, let $f'(c) < 0$, the there exists a $\delta_3 > 0$ such that

$$f(x) < f(c) = m \quad for \quad all \quad x \in (c, c + \delta_3).$$

a contradiction.
If $f'(c) > 0$, then there exists a $\delta_4 > 0$ such that

$$f(x) < f(c) = m \quad for \quad all \quad x \in (c - \delta_4, c)$$

a contradiction. Thus $f'(c) = 0$. This completes the proof.

Intermediate value theorem for derivatives Let f be derivable in $[a, b]$ and $f'(a) \neq f'(b)$, then f' assumes every value between $f'(a)$ and $f'(b)$ i.e., if k lies between $f'(a)$ and $f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = k$.
Proof. With out loss of generality, we assume $f'(a) < f'(b)$ and let $k \in (f'(a), f'(b))$. Define a function g on $[a, b]$ as

$$g(x) = f(x) - kx \quad where \quad x \in [a, b].$$

As f is derivable on $[a, b]$ and kx being linear function is derivable on $\mathbb{R}$ and in particular on $[a, b]$. Thus g being difference of two derivable functions is derivable on $[a, b]$.
Differentiating g, we have

$$g'(x) = f'(x) - k.$$

Now

$$g'(a) = f'(a) - k < 0 \quad and \quad g'(b) = f'(b) - k > 0.$$

Therefore by Theorem E, there exists a point $c \in (a,b)$ such that $g'(c) = 0$ which gives $f'(c) = k$. This completes proof.

Exercise Does there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Hint Apply intermediate value property.

Although derivative of a drivable function need not be continuous (one of the exercises on differentiability) but above theorem (i.e., IVP for derivatives) implies that derivative always has intermediate value property.

Exercise if f is derivable on $[a,b]$. What can we say about range of f'.

Successive Differentiation. Successive differentiation means differentiating a function again and again. Following notations for successive derivatives are commonly used.
Notation. Let y be a function of x. Then its successive derivatives are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}.$$

or

$$y_1, y_2, y_3, \ldots, y_n.$$

or

$$y^1, y^2, y^3, \ldots, y^n.$$

Example. If $y = \frac{\sin^{-1} x}{x}$, then prove that

$$x(1-x^2)y_2 + (2-3x^2)y - 1 - xy = 0.$$

Solution. $y = \frac{\sin^{-1} x}{x} \Rightarrow xy = \sin^{-1} x$
Differentiating w. r. t. x, we have,

$$y + xy_1 = \frac{1}{\sqrt{1-x^2}}$$

Differentiating again w. r. t x, we have

$$y_1 + y_1 + xy_2 = -\frac{1}{2}(1-x^2)^{\frac{-3}{2}}.(-2x) \Rightarrow 2y_1 + xy_2 = \frac{x}{(1-x^2)^{\frac{3}{2}}} \Rightarrow 2y_1 + xy_2 = \frac{1}{1-x^2}\frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow 2(1-x^2)y_1 + x(1-x^2)y_2 = x(y + xy_1) \Rightarrow x(1-x^2)y_2 + (2-3x^2)y_1 - xy = 0.$$

Example If $y = e^{tan^{-1}x}$, then prove that $(1+x^2)y_2 + (2x-1)y_1 = 0$.
Solution. Given that $y = e^{tan^{-1}x}$ so that $y_1 = e^{tan^{-1}x}\frac{1}{1+x^2}$ or $(1+x^2)y_1 = e^{tan^{-1}x} = y$.
Differentiating again w. r. t. x, we get

$$(1+x^2)y_2 + 2xy_1 = y_1 \Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0.$$

Exercises (i). If $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$, then prove that

$$p + \frac{d^2p}{d\alpha^2} = \frac{a^2b^2}{p^3}.$$

(ii). If $y = [x + \sqrt{1+x^2}]^m$, prove that $(1+x^2)y_2 + xy_1 = m^2y$.
(iii). If $y = e^{m\sin^{-1}x}$, prove that $(1-x^2)y_2 - xy_1 = m^2y$.

nth differential coefficients of some functions

(i). To find the nth differential coefficient of x^m, where $m \geq n$.
Let $y = x^m$ so that successive derivatives are given as
$y_1 = mx^{m-1}$, $y_2 = m(m-1)x^{m-2}$,
$y_3 = m(m-1)(m-2)x^{m-3}, \ldots, y_n = [m(m-1)(m-2)\ldots upto\ \ n\ \ factors]x^{m-n}$ or $y_n = [m(m-1)(m-2)\ldots(m-n+1)]x^{m-n}$.
If m is a positive integer and $m = n$ then $y_n = m(m-1)(m-2)\ldots 2.1 = n!$.
(ii). We now find nth derivative coefficient of $(ax+b)^m$.
Let $y = (ax+b)^m$ then
$y_1 = m(ax+b)^{m-1}.a$,
$y_2 = m(m-1)(ax+b)^{m-2}.a^2$,
$y_3 = m(m-1)(m-2)(ax+b)^{m-3}a^3$,
$\ldots, y_n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}a^n$.
If m is a positive integer and $m = n$, then one can simplify to get $y_m = m!a^m$.

(iii). We next find nth derivative of $\frac{1}{ax+b}$, where $x \neq -\frac{b}{a}$.
Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$.
Then $y_1 = -1(ax+b)^{-2}a$,
$y_2 = (-1)^2 2!(ax+b)^{-3}a^2$,
$y_3 = (-1)^3.3.2!(ax+b)^{-4}a^3 = (-1)^3 3!(ax+b)^{-4}a^3$,
$\ldots, y_n = (-1)^n n!(ax+b)^{-(n+1)}a^n$
or $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.

(iv). We next find nth differential coefficient of $y = \sin(ax+b)$.
Let $y = \sin(ax+b)$ so that
$y_1 = a\cos(ax+b)$
$y_1 = a\cos(ax+b = a\sin(ax+b+\frac{\pi}{2}))$ by using $\sin(\frac{\pi}{2}+\theta) = \cos\theta$
$y_2 = a^2\cos(ax+b+\frac{\pi}{2}) = a^2\sin(ax+b+\frac{\pi}{2}+\frac{\pi}{2})$
$= a^2\sin(ax+b+2\frac{\pi}{2})$
$y_3 = a^3\cos(ax+b+2\frac{\pi}{2}) = a^3\sin(ax+b+2\frac{\pi}{2}+\frac{\pi}{2})$
$= a^3\sin(ax+b+3\frac{\pi}{2})$
$\ldots\ y_n = a^n\sin(ax+b+n\frac{\pi}{2})$.
(v). Let us find nth differential coefficient of $e^{ax}\sin(bx+c)$
$y_1 = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$
Put $a = r\cos\theta$, $b = r\sin\theta$ so that $a^2+b^2 = r^2(cos^2\theta + \sin^2\theta) = r^2$ which gives $r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\frac{b}{a}$.
Therefore, $y_1 = e^{ax}[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)] = r.e^{ax}\sin(bx+c+\theta)$.
Similarly, $y_2 = r^2e^{ax}\sin(bx+c+2\theta)$,
$y_3 = r^3e^{ax}\sin(bx+c+3\theta)$
$\ldots, y_n = r^ne^{ax}\sin(bx+c+n\theta)$
Hence, $y_n = (a^2+b^2)^{\frac{n}{2}}e^{ax}\sin[bx+c+n\tan^{-1}\frac{b}{a}]$.

Similarly (i). If $y = \cos(ax+b)$, then $y_n = a^n\cos(ax+b+n\frac{\pi}{2})$
(ii). If $y = a^{mx}$, then $y_n = m^na^{mx}(loga)^n$
(iii). If $y = e^{mx}$, then $y_n = m^ne^{mx}$.
(iv). If $y = e^{ax}\cos(bx+c)$, then $y_n = (a^2+b^2)^{\frac{n}{2}}e^{ax}\cos[bx+c+n\tan^{-1}\frac{b}{a}]$.
(v). If $y = \log(ax+b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
Example. If $y\sqrt{1+x} + x\sqrt{1+y} = 0\ (x \neq y)$, then $\frac{d^ny}{dx^n} = \frac{(-1)^n n!}{(1+x)^{n+1}}$.
Solution. $y\sqrt{1+x} + x\sqrt{1+y} = 0 \Rightarrow y\sqrt{1+x} = -x\sqrt{1+y}$.
Squaring gives $y^2(1+x) = x^2(1+y) \Rightarrow (y^2-x^2) + xy(y-x) = 0$

$\Rightarrow (y-x)(y+x+xy)=0 \Rightarrow y(1+x)=-x \Rightarrow y=\frac{-x}{1+x}=-1+\frac{1}{1+x}$
which gives $y_n=\frac{(-1)^n n!}{(1+x)^{n+1}}$.

Examples (i). Find nth differential coefficient of $\sin^2 x$.
We have $y=\sin^2 x=\frac{1-\cos 2x}{2}=\frac{1}{2}-\frac{1}{2}\cos 2x$
so that $y_n=0-\frac{1}{2}2^n\cos(2x+n\frac{\pi}{2})=-2^{n-1}\cos(2x+n\frac{\pi}{2})$.
(ii). $y=\sin x\sin 3x$
We have $y=\sin x\sin 3x=\frac{1}{2}(2\sin 3x\sin x)=\frac{1}{2}\{\cos 2x-\cos 4x\}$.
So that $y_n=\frac{1}{2}[2^n\cos(2x+n\pi/2)-4^n\cos(4x+n\pi/2)]$.

Leibnitz proved a result which enables us to find nth derivative of product of two functions. $\binom{n}{k}$ denotes the binomial coefficient n choose k. Also following identity holds for binomial coefficients $\binom{m}{r-1}+\binom{m}{r}=\binom{m+1}{r}$.

Leibnitz's Theorem. If $y=uv$, where u and v are functions of x, having derivatives of nth order, then

$$y_n=\binom{n}{0}u_n v+\binom{n}{1}u_{n-1}v_1+\binom{n}{2}u_{n-2}v_2+\cdots+\binom{n}{r}u_{n-r}v_r+\cdots+\binom{n}{n}uv_n.$$

Proof. We shall prove the theorem by Induction on n. For $n=1$, let $y=uv$, we have

$$y_1=u_1v+uv_1=\binom{n}{0}u_1v+\binom{n}{1}uv_1$$

this shows the result is true for $n=1$.

Assume result is true for $n=m$ so that

$$y_m=\binom{m}{0}u_m v+\binom{m}{1}u_{m-1}v_1+\binom{m}{2}u_{m-2}v_2+\binom{m}{r-1}u_{m-r+1}v_{r-1}+\cdots+\binom{m}{r}u_{m-r}v_r+\cdots+\binom{m}{m}uv_m.$$

Differentiating, we have $y_{m+1}=\binom{m}{0}(u_{m+1}v+u_mv_1)+\binom{m}{1}(u_mv_1+u_{m-1}v_2)+\binom{m}{2}(u_{m-1}v_2+u_{m-2}v_3)+\cdots+\binom{m}{r-1}(u_{m-r+2}v_{r-1}+u_{m-r+1}v_r)+\binom{m}{r}(u_{m-r+1}v_r+u_{m-r}v_{r+1})+\cdots+\binom{m}{m}(u_1v_m+uv_{m+1})$

Now $\binom{m}{0}=\binom{m+1}{0}=1$; $\binom{m}{m}=1=\binom{m+1}{m+1}$. Also, $\binom{m}{r-1}+\binom{m}{r}=\binom{m+1}{r}$. Using these, we have

$$y_{m+1}=\binom{m+1}{0}u_{m+1}v+\binom{m+1}{1}u_mv_1+\binom{m+1}{2}u_{m-1}v_2+\cdots+\binom{m+1}{r}u_{m-r+1}v_r+\cdots+\binom{m+1}{m+1}uv_{m+1}.$$

This show that the result is true for $n=m+1$ also. Hence result i true for all $n\geq 1$.

Some applications of Leibnitz Theorem

Examples Using Leibnitz theorem, find nth derivative of following functions.
(i). x^3e^{ax} (ii). $x^2e^x\cos x$
Solution. (i). Choose $u=e^{ax}$ and $v=x^3$. Then

$$u_n=a^ne^{ax},\ u_{n-1}=a^{n-1}e^{ax},\ u_{n-2}=a^{n-2}e^{ax}, u_{n-3}=a^{n-3}e^{ax}.$$

and

$$v_1=3x^2,\ v_2=6x,\ v_3=6,\quad and\quad v_4=0.$$

Now by Leibnitz theorem, we have

$$\frac{d^n}{dx^n}(uv)=\binom{n}{0}u_nv+\binom{n}{1}u_{n-1}v_1+\binom{n}{2}u_{n-2}v_2+\cdots+\binom{n}{n}uv_n$$

so that

$$\frac{d^n}{dx^n}(e^{ax}x^3)=\binom{n}{0}a^ne^{ax}x^3+\binom{n}{1}a^{n-1}e^{ax}.3x^2+\binom{n}{2}a^{n-2}e^{ax}6x+\binom{n}{3}a^{n-3}e^{ax}6$$

$$= e^{ax}a^{n-3}[a^3x^3 + 3na^2x^2 + 6\frac{n(n-1)}{2!}ax + 6\frac{n(n-1)(n-2)}{3!}]$$
$$= e^{ax}a^{n-3}[a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)].$$

(ii). Let $u = e^x \cos x$ and $v = x^2$, we have
$u_n = r^n e^x \cos(x + n\theta),\ u_{n-1} = r^{n-1}e^x \cos[x + (n-1)\theta],\ u_{n-2} = r^{n-2}e^x \cos[x + (n-2)\theta]$
$where \ \ r = \sqrt{2} \ \ and \ \ \theta = \frac{\pi}{4}$,
and

$$v_1 = 2x, v_2 = 2 \ \ and \ \ v_3 = 0.$$

By Leibnitz theorem

$$\frac{d^n}{dx^n}(e^x \cos x.\, x^2) = \binom{n}{0}2^{\frac{n}{2}}e^x \cos(x+n\frac{\pi}{4})x^2 + \binom{n}{1}2^{\frac{n-1}{2}}e^x \cos(x+(n-1)\frac{\pi}{4}).2x + \binom{n}{2}2^{\frac{n-2}{2}}e^x \cos(x+(n-2)\frac{\pi}{4})2$$
$$= 2^{\frac{n}{2}}e^x[x^2 \cos(x + n\frac{\pi}{4}) + \sqrt{2}nx \cos(x + (n-1)\frac{\pi}{4}) + \frac{n(n-1)}{2}\cos(x + (n-2)\frac{\pi}{4})].$$

Examples continued.
(iii). Let $y = \sin(m \sin^{-1} x)$. Show that

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - m^2)y_n = 0.$$

Solution. $y = \sin(m \sin^{-1} x)$ so that $y_1 = \cos(m \sin^{-1} x)\frac{m}{\sqrt{1-x^2}}$.
This gives

$$(1 - x^2)y_1^2 = m^2[1 - \sin^2(m \sin^{-1} x)]$$

or

$$(1 - x^2)y_1^2 - m^2(1 - y^2) = 0.$$

Differentiating, we have

$$2(1 - x)^2 y_1 y_2 - 2xy_1^2 + 2m^2 yy_1 = 0.$$

Dividing both sides by $2y_1$, we have

$$(1 - x^2)y_2 - xy_1 + m^2 y = 0$$

Differentiating n times and applying Leibnitz theorem, we have

$$[y_{n+2}(1 - x^2) - 2nxy_{n+1} + \frac{n(n-1)}{2}y_n(-2)] - [xy_{n+1} + ny_n] + m^2 y_n = 0$$

or

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (-n^2 + n - n + m^2)y_n = 0$$

or

$$(1 - x^2)y_{n+2} - 2(2n + 1)xy_{n+1} - (n^2 - m^2)y_n = 0$$

or

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - m^2)y_n = 0.$$

(iv). If $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that

$$y_n = (n - 1)^2 y_{n-2} \ \ for \ \ x = 0.$$

Solution.

$$y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \Rightarrow y\sqrt{1 - x^2} = \sin^{-1} x.$$

Differentiating w.r.t. x, we get

$$\sqrt{1-x^2}y_1 + \frac{-2xy}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \quad or \quad (1-x^2)y_1 - xy = 1.$$

Differentiating n times and using Leibnitz theorem, we have

$$[y_{n+1}(1-x^2) + -2xny_n - 2\frac{n(n-1)}{2}y_{n-1}] - [xy_n + ny_{n-1}] = 0$$

or

$$(1-x^2)y_{n+1} - 2nxy_n - (n^2-n)y_{n-1} - xy_n - ny_{n-1} = 0.$$

or

$$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0.$$

changing n to $n-1$, we have

$$(1-x^2)y_n - (2n-1)xy_{n-1} - (n-1)^2y_{n-2} = 0.$$

Putting $x = 0$, we have

$$y_n(0) = (n-1)^2y_{n-2}(0)$$

(v). If $y = (x+\sqrt{1+x^2})^m$, find $y_n(0)$.
One can show

$$y = \frac{my}{\sqrt{1+x^2}}$$

$$and \quad (1-x^2)y_{n+2} + (2n+1)xy_{n+1} - (m^2-n^2)y_n.$$

Putting $n = 1, 2, 3, 4, \ldots$, we have
$y_3(0) = (m^2-1^2)y_1(0) = m(m^2-1^2)$, $y_4 = (m^2-2^2)y_2(0) = m^2(m^2-2^2)$, $y_5(0) = (m^2-3^2)y_3(0) = m(m^2-1^2)(m^2-3^2)$ etc.
Therefore, when n is odd,

$$y_n(0) = m(m^2-1^2)(m^2-3^2)\ldots[m^2-(n-2)^2].$$

and when n is even,

$$y_n(0) = m^2(m^2-2^2)(m^2-4^2)\ldots[m^2-(n-2)^2].$$

Exercises on Leibnitz theorem.
(i). If $y = x\log x$, then prove that for $n > 1$

$$y_n = (-1)^n\frac{(n-2)!}{x^{n-1}}.$$

(ii). If $y = x^{n-1}\log x$, then prove that $xy_n = (n-1)!$.
(iii). If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, then prove that

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-x^2)y_n = 0$$

.

Partial differentiation.
Limits of functions of two variables. Let $f(x,y)$ be defined in a neighbourhood of a point (a,b) with

the possible exception of the point (a, b) itself. Then the function $f(x, y)$ is said to tend the limit L as $x \to a, y \to b$, if for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$0 < |x - a| < \delta \ \ and \ \ 0 < |x - b| < \delta \Rightarrow |f(x, y) - L| < \epsilon.$$

or some authors write

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon.$$

Here we note that we are not concerned with the value of the function $f(x, y)$ at the point (a, b), it need not be even defined at (a, b). This notion of limit is called simultaneous limit. There is concept of repeated limit also and we ignore that at this moment.

Continuity. A function $f(x, y)$ is said to be continuous at point (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. In terms of $\epsilon - \delta$, we say $f(x, y)$ is continuous at point (a, b) if given $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - f(a, b)| < \epsilon.$$

Partial derivatives. Let $z = f(x, y)$ be a continuous function of two independent variables x and y. If we keep y constant and vary x alone, then z becomes a continuous function of x alone. If this function possesses a derivative (ordinary), this derivative is called the partial derivative of z with respect x. This partial derivative is denoted by various symbols such as $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x, D_x f(x, y)$ etc.

Definition. The partial derivative of $f(x, y)$ w.r.t. x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \frac{d}{dx} f(x, y_0)|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exits.

Examples (i). Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution. To find $\frac{\partial f}{\partial x}$, we regard y as a constant and differentiate w. r. t. x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3.1.y + 0 - 0 = 2x + 3y.$$

the value of $\frac{\partial f}{\partial x}$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\frac{\partial f}{\partial y}$, we regard x as constant and differentiate w. r. t. y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3.x.1 + 1 - 0 = 3x + 1.$$

The value of $\frac{\partial f}{\partial y}$ at $(4, -5)$ is $3(4) + 1 = 13$.

(ii). Find $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin xy$.

Solution. We treat x as a constant and f as a product of y and $\sin xy$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y} \sin xy + \sin xy \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Partial derivatives of higher order. The first partial derivatives of $z = f(x, y)$ are themselves function of x and y and can therefore be differentiated partially with respect to x or y. The four derivatives thus obtained, called the second order partial derivatives of z or $f(x, y)$ are

$$\frac{\partial}{\partial x}(\frac{\partial z}{\partial x}), \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}), \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}), \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

and are denoted as

$$f_{xx}(x,y), f_{yx}(x,y), f_{xy}(x,y), f_{yy}(x,y)$$

or simply as

$$f_{xx}, f_{yx}, f_{xy}, f_{yy}.$$

The distinction between $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ should me noted carefully. The first is obtained by first differentiating z partially with respect to x and then differentiating the result so obtained partially with respect to y, whereas the second is obtained by performing the two operations of differentiation in the reverse order. The two derivatives $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ are not equal in general.

Example Find all the second order partial derivatives of z, where $z = \log(e^x + e^y)$.
Solution. Here $\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y}$, $\frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y}$
Now $\frac{\partial^2 z}{\partial^2 x} = \frac{(e^x+e^y)e^x - e^x.e^x}{(e^x+e^y)^2} = \frac{e^x.e^y}{(e^x+e^y)^2}$
$\frac{\partial^2 z}{\partial x \partial y} = \frac{-e^x e^y}{(e^x+e^y)^2} = \frac{\partial^2 z}{\partial y \partial x}$
$\frac{\partial^2 z}{\partial^2 y} = \frac{(e^x+e^y)e^y - e^y.e^y}{(e^x+e^y)^2} = \frac{e^y.e^x}{(e^x+e^y)^2}$.

Example. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.
Solution.

$$u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}.$$

so that

$$\frac{\partial u}{\partial y} = x^2.\frac{1}{1+(\frac{y}{x})^2}.\frac{1}{x} - 2y\tan^{-1}\frac{x}{y} - y^2.\frac{1}{1+(\frac{x}{y})^2}.(\frac{-x}{y^2}).$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} = x - 2y\tan^{-1}\frac{x}{y}.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}[\frac{\partial u}{\partial y}] = \frac{\partial}{\partial x}(x - 2y\tan^{-1}\frac{x}{y}) = 1 - 2y.\frac{1}{1+(\frac{x}{y})^2}.\frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Example. Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions.

$$(i).u = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \quad (ii).u = e^{ax}\cos by.$$

Solution. (i). $\frac{\partial u}{\partial x} = 3ax^2 + 6bxy + 3cy^2$, $\frac{\partial^2 u}{\partial y \partial x} = 6bx + 6cy$.
and $\frac{\partial u}{\partial y} = 3bx^2 + 6cxy + 3dy^2$ and $\frac{\partial^2 u}{\partial x \partial y} = 6bx + 6cy$.
Hence $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.
(ii). $\frac{\partial u}{\partial x} = ae^{ax}\cos by$; $\frac{\partial^2 u}{\partial y \partial x} = -abe^{ax}\sin by$.
$\frac{\partial u}{\partial y} = -be^{ax}\sin by$; $\frac{\partial^2 u}{\partial x \partial y} = -abe^{ax}\sin by$
Hence .....

Exercises. (i). If $z = \frac{1}{2}\log(x^2+y^2)$ or $\tan^{-1}(\frac{y}{x})$, prove that
(a). $(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = \frac{1}{x^2+y^2}$
(b). $(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = 0$

(ii). If $u = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

(iii). If $u = \sin^{-1} \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

Total differentials. Let $u = f(x, y)$ be a function of two variables x and y. Let $\delta x, \delta y$ be increments of x, y and δu be corresponding increment in u. Then

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \epsilon_1 \delta x + \epsilon_2 \delta y.$$

When δx and $\delta y \to 0$, δu also tends to zero. Since $\epsilon_1, \epsilon \to 0$ as $\delta x, \delta y \to 0$ and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are finite quantities. The part $\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$ and is also known as the total differential of u with respect to x and y and is denoted by du.
Hence

$$du = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y.$$

If $u = x^2y + xy^2$ so that $du = (2xy + y^2)dx + (x^2 + 2xy)dy$.

Homogeneous functions. A function $f(x, y)$ of two variables x and y is said to be homogeneous function of degree or order n if, for all values of t

$$f(tx, ty) = t^n f(x, y).$$

Thus if $u = ax^2 + 2hxy + by^2$, then u is homogeneous function of degree 1, for

$$a(tx)^2 + 2h(tx)(ty) + b(ty)^2 = t^2(ax^2 + 2hxy + by^2).$$

Again if $u = x\tan^{-1}(\frac{y}{x})$, then u is homogeneous function of degree 1, for

$$(tx)\tan^{-1}(\frac{ty}{tx}) = tx\tan^{-1}(\frac{y}{x}).$$

This definition can be extended to a function of any number of variables.

Exercise. Check whether following functions are homogeneous or not.
(i). $\frac{x^3+y^3}{x-y}$.
(ii). $\sqrt{(xy)}\sin^{-1}\frac{y}{x}$.

Euler's theorem on homogeneous functions. Let u be a homogeneous function of degree n in x and y, and possess continuous partial derivatives, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Since u is a homogeneous function of degree n in x and y, therefore

$$u = x^n f(\frac{y}{x}) \Rightarrow \frac{\partial u}{\partial x} = nx^{n-1}f(\frac{y}{x}) + x^n.f'(\frac{y}{x}).(-\frac{y}{x^2}) = nx^{n-1}f(\frac{y}{x}) - yx^{n-2}f'(\frac{y}{x}).$$

or

$$x\frac{\partial u}{\partial x} = nx^n f(\frac{y}{x}) - yx^{n-1}f'(\frac{y}{x}) = nu - yx^{n-1}f'(\frac{y}{x}).$$

Also,

$$\frac{\partial u}{\partial y} = x^n.f'(\frac{y}{x}).\frac{1}{x} = x^{n-1}f'(\frac{y}{x}) \Rightarrow y\frac{\partial u}{\partial y} = yx^{n-1}f'(\frac{y}{x})$$

Therefore

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

We note that Euler's theorem can be extended to a homogeneous of any finite number of variables. Let $f(x_1, x_2, \ldots, x_n)$ be a homogeneous function of degree m in n variables $x_1, x_2, \ldots, x_n$. Then

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n} = mf.$$

Example (i). If $u = f(\frac{y}{x})$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
Solution. Here u is homogeneous of degree 0. Apply Euler's theorem, the result follows.

Example (ii). If $V = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + \frac{1}{2}\cot V = 0.$$

Solution.

$$V = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow \cos V = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+\frac{y}{x})}{\sqrt{x}(1+\sqrt{\frac{y}{x}})} = x^{\frac{1}{2}} f(\frac{y}{x}).$$

Therefore, $\cos V$ is a homogeneous function of degree $\frac{1}{2}$ in x and y. By Euler's Theorem, we have

$$x\frac{\partial}{\partial x}(\cos V) + y\frac{\partial}{\partial y}(\cos v) = \frac{1}{2}\cos V \Rightarrow x(-\sin V)\frac{\partial V}{\partial x} + y(-\sin V)\frac{\partial V}{\partial y} = \frac{1}{2}\cos V$$

$$\Rightarrow x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + \frac{1}{2}\cot V = 0.$$

Example (iii). If $u = \sin^{-1} \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$.
Solution.

$$u = \sin^{-1} \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \Rightarrow \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{(1-\sqrt{\frac{y}{x}})}{1+\sqrt{\frac{y}{x}}} = x^0 f(\frac{y}{x}).$$

Therefore $\sin u$ is a homogeneous function in x and y of degree 0.
By Euler's Theorem, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 0.\sin u \Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0 \;\; \textit{which gives} \;\; \frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

Exercises. (i). If $u = x^n f(\frac{y}{x})$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \;\; \textit{and} \;\; x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

(ii). If $z = xyf(\frac{y}{x})$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

(iii). If $u = \tan^{-1} \frac{x^2+y^2}{x+y}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u.$$

(iii). Prove Euler's Theorem for functions of three variables.

For Graphics to understand $\epsilon - \delta$ definition of limits and continuity, I suggest students should read [3]. Without graphics, I think it is difficult to understand. Solution manual of [3] is also available in market. If you find any typing mistakes or errors in proofs of theorems or solutions of examples, please send me an email at any of the following email address, so that I can correct them.
mushtaqab1125@gmail.com or mushtaqab@math.iitb.ac.in

Acknowledgements. I would like to thank Reyaz Ahmad Bhat (IT Section SAM Degree College Budgam) for providing Internet and other facilities.

Mushtaq Ahmad Bhat.

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